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ELEMENTS

OF

WAVE MOTION

RELATING TO

SOUND AND LIGHT.

ROBERT DRUMMOND, ELECTROTYPER AND PRINTER, NEW YORK.

PREFACE.

THIS text-book, as is stated on the title-page, has been prepared expressly for the use of the Cadets of the United States Military Academy, and this specific object has therefore wholly controlled its design and restricted its scope. It is thus in no sense a treatise. Because of the limited time allotted to the subjects of sound and light in the present distribution of studies at the Academy, the problem of arranging a fundamental course of sufficient strength, to be something more than popular, and yet to be mastered within the allotted time, has been somewhat perplexing. The basis of this arrangement is necessarily the mathematical attainments of the class for which the course is intended. In this respect, the class has completed the study of elementary Mathematics, as far as to include the Calculus, and has had a four months' study of the application of pure Mathematics, in a course of Analytical Mechanics. With these elements to govern, this text-book has been designed for a seven weeks' course, including advance and review. The fact of being able, through the discipline of the Academy, to exact of each student a certain number of hours of hard study on each lesson, is of course an important element necessary to be stated.

The study of the text is supplemented by lectures, in which the

principles of Acoustics and Optics are amply illustrated by the aid of a very well equipped laboratory of physical apparatus. Carefully written notes of the lectures are submitted by each student to the instructor on the following morning for revision and criticism. Important errors of fact and misinterpretation of principle are thus at once detected, corrected, and hence prevented from obtaining a lodgment in the mind of the pupil. Another element in this matter of instruction, of sufficient importance to be mentioned, is the opportunity freely exercised by each student of making known the difficulties that he has encountered, before being called upon to exhibit his proficiency in the lesson of the day. It is required that these difficulties shall be clearly and exactly stated, in order that the instructor may, by a judicious question or a concise explanation, enable the student to clear up the difficulty as of himself, and thus complete the elucidation.

The author believes that this method of instruction, taken as a whole, is in sufficiently intimate accord with the text as to gain the following advantages, viz.: 1°, the tasks are of the requisite strength to demand all the study-time allotted to his department of instruction, and thus is secured the invaluable mental effort and discipline due to a specified number of hours of hard study; 2°, while the daily tasks are progressive, they are based on fundamental principles which require the exercise of a rational faith, and develop a continual growth of confidence in the mind of the pupil, and a belief in his own ability to overcome each difficulty as it arises; 3°, when the course is completed, the student finds himself equipped with a satisfactory knowledge of the essential principles of the physical science, to which he may add by further individual study, without the necessity of reconstructing his foundation.

The elements of character developed in the student by this system of instruction, viz., confidence in his powers, reliance on individual effort, and capacity to appreciate truly his sources of information, are of essential importance in a career where he may be called upon in emergencies to exercise self-control, and to meet manfully unforeseen difficulties; and they offer a sufficient reason for the importance given to these studies in the curriculum of the Academy.

Text-books are generally compilations. The subject-matter of this text has been gathered by the author from whatever source appeared to him best for the purpose in view. And as it is often desirable to refer to original treatises, for a better conception of the subject under discussion, a list of authors is appended to this Preface.

In the arrangement of the matter, the author has been governed alone by the necessities of the case and the restrictions of the course. It has therefore seemed advisable to arrive at the deduction of Fresnel's wave surface as expeditiously as possible, and on the way to establish all of the essential principles of undulatory motion common to sound and light. Sufficient theoretical attention is paid in the text to the wave surface, and a study of its model in the lecture-room makes clear its important properties and those of its special cases. Acoustics is briefly treated, and is indeed made subsidiary to Optics, by utilizing its numerous illustrations in vibratory motion, so that the laws of this motion may be the more clearly apprehended in the subject of light. In Optics, while the essential principles of the deviation of light by lenses and mirrors, the construction of optical images and the principal telescopic combinations, are carried only to first approximations, and are some-

what more condensed than is usual, nothing essential to the Academic course of Astronomy has been omitted. The part relating to physical Optics is very concise, but the experiments performed and illustrations given in the lecture room, especially in diffraction, dispersion, and polarization, largely remedy this defect.

The figures throughout the text were drawn by Lieut. Arthur Murray, 1st U. S. Artillery, Acting Asst. Professor of Philosophy, U. S. M. A., to whom I desire to acknowledge my great indebtedness.

P. S. M.

West Point, N. Y., May, 1882.

LIST OF AUTHORITIES.

1. Airy. Undulatory Theory of Optics. 1866.
2. Annales de Chimie et de Physique. Tome XXXVI.
3. Archives de Musée Teyler. Vol. I. 1868.
4. Bartlett. Analytical Mechanics. 1858.
5. Billet. Traité d'Optique Physique. 1859.
6. Challis. Lectures on Practical Astronomy. 1879.
7. Coddington. System of Optics. 1829.
8. Comptes Rendus. Vol. LXVI. 1863.
9. Daguin. Traité de Physique. Vols. I and IV. 1879.
10. Donkin's Acoustics. 1870.
11. Encyclopedia Britannica. Ninth Edition, Vol. VII. 1877.
12. Everett's Units and Physical Constants. 1879.
13. Fresnel. Œuvres de. Vols. I, II.
14. Helmholtz. Tonempfindungen. 1866.
15. Helmholtz. Physiologische Optik. 1867.
16. Jamin. Cours de Physique. Vol. III. 1866.
17. Lamé. Théorie Mathématique de l'Elasticité. 1852.
18. Lloyd. Wave Theory of Light. 1873.
19. Parkinson's Optics. Third Edition. 1870.
20. Potter's Optics. 1851.
21. Price. Infinitesimal Calculus. Vol. IV. 1865.
22. Rood. Modern Chromatics. 1879.
23. Thomson and Tait. Natural Philosophy. Vol. I. 1879.
24. Verdet. Cours de Physique. Vol. II. 1869.
25. Verdet. Leçons de l'Optique Physique. Vols. I and II. 1869.

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PART I.

WAVE MOTION.

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1. Equation (E) of Analytical Mechanics (Michie),

$$\Sigma I\delta p - \Sigma m \frac{d^2s}{dt^2}\delta s = 0,$$

expresses in mathematical language the law that the potential energy expended is equal to the kinetic energy developed. Every analytical discussion of the action of force upon matter must be founded upon this general equation. For the complete solution of every problem of energy, it is necessary to know the intensities, lines of action, and points of application of the acting forces, the masses acted upon, and to possess a perfect mastery of such mathematical processes as are necessary to pass to the final equations whose interpretation will make known the effects. These difficulties, which, in Mechanics, limit the discussion to the free, rigid solid, and to the perfect fluid, are, in Molecular Mechanics, almost insuperable; since we neither know the nature of the forces which unite the elements of a body into a system, nor the constitution of the elements themselves.

2. But the faculty of observation, being cultivated and logically directed, has enabled scientific men to originate experiments which, because of our inherent faith in the uniformity of the laws of nature, have resulted in certain hypotheses as to the nature of sound, light, heat, and other molecular sciences. When an hypothesis not only satisfactorily explains the known phenomena of the science in question, but even predicts others, it then becomes a theory, and its acceptance is more or less complete. An hypothesis is related to a theory as the scaffolding to the structure, the latter being so proportioned in all its parts as to be in the completest harmony, while the former may be modified in any way to suit the ever-varying necessities of the architect.

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While there are many matters concerning which a reasonable doubt may be entertained, because of insufficient data, the progress of scientific thought and the fertility of scientific research have, within recent times, established certain facts that are now universally accepted.

3. ***Molecular Science.*** Molecular science is a branch of Mechanics in which the forces considered are the attractions and repulsions existing among the molecules of a body, and the masses acted upon are the indefinitely small elements, called molecules, of which the body is composed. It embraces light, heat, sound, electrics, and, in one sense, chemistry.

4. From the facts of observation and experiment, it is assumed that all matter, whether solid, liquid, or gaseous, is made up of an innumerable number of molecules in sensible, though not in actual contact; that these molecules are so small as not to be within range of even our assisted vision; and that they are separated from each other by distances which are very great compared with their actual linear dimensions.

5. The molecular forces, which determine the particular state of the matter, are either attractive or repulsive. When the attractive forces exceed the repulsive in intensity, the body is a solid; when equal to the repulsive, a liquid; and when less, a gas. The relative places of equilibrium of the molecules are determined by the molecular forces called into play by the action of extraneous forces applied to the body. Thus, when a solid bar is subjected to the action of an extraneous force, either to elongate or to compress it, the molecules assume new positions of equilibrium with each increment of force, and, in either case, the aggregate molecular forces developed are equal in intensity, but contrary in direction, to the extraneous force applied. In general, where rupture does not ensue, the extraneous forces applied are much less than the molecular forces capable of being called into play.

6. While we are ignorant of the true nature of force and matter, our senses enable us to appreciate the effects of the former upon the latter. Our whole knowledge of the physical sciences is based upon the correct interpretation of these sensuous impressions. Observation teaches that if a body be subjected to the action of an extraneous force, the effect of the force is transmitted throughout

the body in all directions, and since the body is connected with the rest of the material universe, there is no theoretical limit to the ultimate transfer of this effect throughout space.

7. Among the appreciable effects of force are the changes of state with respect to rest and motion. These can be transferred from an origin to another point in but two ways, viz.:

1°. By the simultaneous transfer of the body, which is the depository of the motion.

2°. By the successive actions and reactions between the consecutive molecules along any line from the origin.

In the molecular sciences, the latter is assumed to be the method of transfer, and the object of the succeeding discussion is to investigate the nature of the disturbance, the circumstances of its progress, and the behavior of the molecules as they become involved in it.

8. While the initial disturbance is perfectly arbitrary, the molecular motions produced through its influence in any medium are necessarily subjected to the variable conditions which result from the action of the forces that unite the molecules into a material system. The problems are then those of constrained motion.

9. Among the physical properties of bodies, elasticity is of such great importance, that a complete knowledge of its mathematical theory is essential to the thorough elucidation of many of the phenomena of molecular science. The limits of this text permit but a passing allusion to its more important laws.

ELASTICITY.

10. A body is said to be homogeneous when it is formed of similar molecules, either simple or compound, occupying equal spaces, and having the same physical properties and chemical composition. In such a body, a right line of given length l and determinate direction is understood to pass through the same number n of molecules wherever it is placed; the ratio $\frac{l}{n}$ will vary with the direction of l. In crystalline bodies, considered as homogeneous, $\frac{l}{n}$ varies with the direction; in homogeneous non-crystalline bodies, such as glass, the ratio varies insensibly, or is independent of the

direction. This supposition requires n to be very great, however small l may be.

11. That property, by which the internal forces of a body or medium restore, or tend to restore, the molecules to their primitive positions, when they have been moved from these positions by the action of some external force, is called *Elasticity*.

12. The elasticity is said to be perfect when the body always requires the same force to keep it at rest in the same bulk, shape, and temperature, through whatever variations of bulk, shape, and temperature it may have been subjected.

13. Every body has some degree of elasticity of bulk. If a body possess any degree of elasticity of shape, it is called a solid; if none, a fluid. All fluids possess great elasticity of bulk. While the elasticity of shape is very great for many solids, it is not perfect for any. The degree of distortion within which elasticity of shape is found, is essentially limited in every solid; when the distortion is too great, the body either breaks or receives a permanent *set;* that is, such a molecular displacement that it does not return to its original figure when the distorting force is removed.

14. The limits of elasticity of metal, stone, crystal, and wood are so narrow that the distance between any two neighboring molecules of the substance never alters by more than a small proportion of its own amount, without the substance either breaking or experiencing a permanent set. In liquids, there are no limits of elasticity as regards the magnitude of the positive pressures applied; and in gases, the limits of elasticity are enormously wider with respect to rarefaction than in either solids or liquids, while there is a definite limit in condensation when the gas is near the critical temperature.

15. The substance of a homogeneous solid is called *isotropic* when a spherical portion exhibits no difference, in any direction, in quality, when tested by any physical agency. When any difference is thus manifested, it is said to be *æolotropic.*

16. ***Origin of the Theory of Elasticity.*** In Mechanics, by supposing the bodies perfectly rigid, and the distances of the points of application of the extraneous forces invariable, however great the forces, the problems are much simplified, without affecting their generality. But this ignores the law by which the

reciprocal influence is transmitted from point to point of the body, and by which the action of one force is counterbalanced by the actions of others. In reality, the body undergoes deformation, and when the limit is reached, rupture ensues. The mathematical theory has arisen from the necessity of a knowledge whereby these permanent deformations and rupture may be avoided. This theory has been extended to the determination of the laws of small motions, or, in general, to the vibrations of elastic media.

17. The initial state of a homogeneous body is considered to be that in which it is perfectly free from all extraneous forces, to be, indeed, that of a body falling freely in vacuo. Such a body is then the geometrical place of an innumerable number of material points, which are distinguished from the rest of space by several mechanical properties. Each of these material points is called a molecule.

18. When such a body is subjected to the action of an extraneous force, either a tension or a pressure, a motion of its surface particles ensues, and this disturbance is propagated to the interior molecules ; the body becomes slightly distorted, and soon takes a new state of equilibrium. When the external forces are removed, the internal forces are again balanced, and the original condition is restored, provided there is no permanent set. All changes of form of a solid, or any variation of the relative distances of its material points, are ever accompanied by the development of attractive or repulsive forces between the molecules. These variations and forces begin, increase, decrease, and end at the same time, and hence are mutually dependent.

19. The properties of a solid body depending only upon those of its material points, they alone are the foci whence emanate these interior forces.

20. Let an extraneous force be applied to a body, and consider its effect upon any two molecules sufficiently near each other to be mutually affected by their changes of position. Should one of the molecules, on account of this exterior action, approach the other, a mutual repulsion takes place, which, in time, overcomes the motion of the first molecule, and causes the second to take its new position of equilibrium with respect to the first. The reverse is the case when the first molecule withdraws from the second, and an attractive force is developed between them. If r represent the primitive distance, Δr may represent the displacement. Then the intensity

of the attractive or repulsive force developed between the molecules may be represented by $f(r, \Delta r)$. This function becomes zero when Δr is zero, whatever r may be; it decreases rapidly when r has a sensible value, whatever Δr may be, since all cohesion ceases between two parts of the same body separated by an appreciable distance. Assuming that the intensity of the molecular forces varies directly with the degree of displacement, this limitation embodies only the cases where the changes of form are very small, whether the extraneous forces are extremely small or the bodies considered have great rigidity. Hence, $f(r, \Delta r)$ is limited to the product of a function of r and the first power of Δr, which becomes infinitely small when Δr becomes infinitely small.

21. *Elastic Force defined.* From any molecule M in the interior of a solid, with a radius equal to the greatest distance beyond which $f(r)$ is insensible, describe a sphere. This volume will embrace all molecules that influence the molecule M, and may be called the *sphere of molecular activity*. Pass a plane through M, dividing the sphere into the two parts SAC and SBC. Normal to LN and having for its base a differential surface ω, conceive a cylinder in the hemisphere SBC. When the equilibrium is disturbed, the molecules in SAC will act on the molecules of the cylinder. The resultant ωE of all these actions is called the *elastic force* exerted by SAC upon SBC, referred to the infinitesimal surface ω. Integrating this function with respect to the plane, we obtain the elastic force referred to the circle SMC. The resultant ωE will, in general, be oblique to the plane element ω. If it is normal to this element and directed towards the hemisphere SAC, it will be a traction; if normal and directed toward SBC, it will be a pressure; if parallel to the plane SMC, it will be the tangential elastic force.

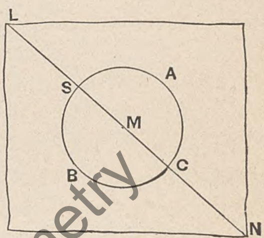

Figure I.

Similarly, if the cylinder is situated in the hemisphere SAC, the resultant elastic force exerted upon the molecules of the cylinder by the molecules in SBC is represented by $\omega E'$, referred to the same elementary surface ω. If the body, slightly changed in form,

is in equilibrium of elasticity, the two elastic forces ωE and $\omega E'$ should be equal in intensity, but contrary in direction. Both, however, will represent either pulls, pressures, or tangential forces; that is, if one is a pull, the other will be a pull directly opposed to it.

The elastic force ωE, considered with reference to the element planes ω drawn parallel to each other through all points of the body, will vary in intensity and direction from point to point; and at the same point M will vary with the orientation of the element plane ω.

22. The direction of the planes ω may be determined by that of their normals. Using the angles ϕ and ψ to designate the latitude and longitude of the point where the normal pierces the surface of the sphere of activity, and representing by x, y, and z the co-ordinates of this point referred to the co-ordinate axes, we have

$$x = \cos\phi \cos\psi, \qquad y = \cos\phi \sin\psi, \qquad z = \sin\phi.$$

Representing the orthographic projections of ωE by ωX, ωY, and ωZ upon the co-ordinate axes, we see, in the case of equilibrium of elasticity, that ωE will be a function of the five variables x, y, z, ϕ, and ψ; and if the motion be progressive, the variable t will also enter. X, Y, and Z can be determined from ωE, ϕ, and ψ; and, reciprocally, the latter from the former. X, Y, and Z are, however, usually determined, and are, in general, functions of the six variables (x, y, z, ϕ, ψ, t), and which being found according to the special circumstances that cause the deformation of the body, would enable us to ascertain, at each instant and at each point of the body, the direction and intensity of the elastic force exerted upon every element plane passing through the given point. In brief, the determination of these functions and the study of their properties are the principal objects of the mathematical theory of elasticity.

23. ***Elasticity of Solids.*** Experiment has shown that, when a solid bar is subjected to small elongations, or those within elastic limits, the following laws are verified, viz.: 1°, the elongations are directly proportional to the length of the bar; 2°, they are inversely proportional to the area of cross section; 3°, they are directly proportional to the intensity of the elongating force; 4°,

they are variable for bars of different materials. These experimental laws can be expressed by the equation,

$$\lambda = \frac{1}{M} \cdot \frac{Pl}{s}, \tag{1}$$

in which l is the length of the bar unloaded, s the area of cross-section, P the intensity of the stretching force, M a coefficient varying with the nature of the material, and λ is the corresponding elongation. Making $s = 1$, $\lambda = l$, we get, from the above equation, $P = M$. If, therefore, the law of the elongation should remain true for all intensities, M would be that intensity which, applied to a bar of unit area in cross-section, would make the elongation equal to the original length. Such an hypothesis gives us the value of the coefficient M, which can be used within the limits of experiment. M is called the *coefficient* or *modulus* of longitudinal elasticity, or Young's modulus. While we cannot experiment over such wide limits in longitudinal compression, because of the liability to flexure, the same laws are held to be applicable, with the same limitations. Taking the metre for the unit of length, the square centimetre for the unit of area, and the gramme for the unit of intensity, the moduli of longitudinal elasticity for the principal metals are, according to Wertheim, as follows:

Lead,	177×10^6	Copper,	1245×10^6
Gold,	813×10^6	Platinum,	1704×10^6
Silver,	736×10^6	Iron,	1861×10^6
Zinc,	873×10^6	Steel,	1955×10^6

The coefficient of elasticity decreases with increase of temperature between 15° and 200° C.

24. An isotropic solid has, in addition to the modulus of longitudinal elasticity, a modulus of rigidity; the former relating to the elasticity of bulk or volume, and the latter to that of shape. If a bar be of square cross-section before elongation, it will be found afterwards to have undergone deformation in its angles, although the diagonals of the cross-section may still be at right angles. The numerical ratio of the intensity of the force applied, to the deformation produced is the *modulus of rigidity*. The deformation is measured by the change in each of the four right angles, in terms of the radian (57°.29) as unity.

25. *Fundamental Coefficients of Elasticity.* Let there be a rectangular parallelopipedon AH, subjected at first to the action of equal and opposite normal pressures on the two bases AD and EH. The vertical edges will, by the laws of elongation, shorten, and the horizontal edges increase in length; and the relative changes in length will be proportional to the quotient of the normal pressures by the area AD; that is, to the pressure on the unit of area.

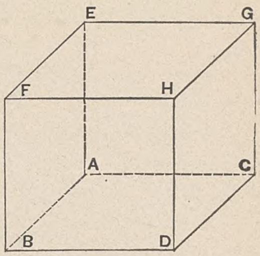

Figure 2.

Let α be the relative shortening of the vertical edges, β the relative increase of the horizontal edges, and P the pressure on the unit of area, then

$$\alpha = mP, \qquad \beta = nP,$$

m and n being coefficients to be determined only by experiment. If Q be the pressure applied to the unit area on the faces AF and CH, the edge AC will be shortened α', and the edges AB, AE lengthened β', and we will have

$$\alpha' = mQ, \qquad \beta' = nQ.$$

If R be the pressure on the unit area of the faces AG and BH, the edge AB will be shortened α'', and the edges AE and AC elongated β'', and we will have

$$\alpha'' = mR, \qquad \beta'' = nR.$$

If now the three pairs of pressure, P, Q, R, act simultaneously, their effects will be superposed, and, representing by ε, ε', ε'', the relative variations of the lengths of the edges AE, AC, and AB, we will have

$$\left.\begin{aligned} \varepsilon &= \alpha - (\beta' + \beta'') = mP - n(Q + R), \\ \varepsilon' &= \alpha' - (\beta + \beta'') = mQ - n(P + R), \\ \varepsilon'' &= \alpha'' - (\beta + \beta') = mR - n(P + Q); \end{aligned}\right\} \tag{2}$$

from which we readily deduce,

$$\left.\begin{aligned} P &= H\varepsilon + K(\varepsilon' + \varepsilon''), \\ Q &= H\varepsilon' + K(\varepsilon + \varepsilon''), \\ R &= H\varepsilon'' + K(\varepsilon + \varepsilon'); \end{aligned}\right\} \tag{3}$$

in which

$$\left.\begin{aligned} H &= \frac{m-n}{m(m-n)-2n^2}, \\ K &= \frac{n}{m(m-n)-2n^2}. \end{aligned}\right\} \tag{4}$$

Hence the pressures exerted upon the faces of the volume, and therefore the elastic reactions, can be expressed as linear functions of the relative variations of the length of the edges by means of two constant coefficients. These two coefficients, H and K, are fundamental in the theory of elasticity. They can only be determined by experimental investigations; once determined for any body, the problems of elasticity become those of rational mechanics.

Exact analysis of the conditions of equilibrium in the interior of a solid elastic body shows that, in each point of the body, there exist three rectangular directions, variable from one point to another, such that the elements perpendicular to these directions support normal pressures or tractions.

An infinitely small parallelopipedon, having its edges parallel to these three directions, is in the condition of that discussed above; and it suffices to express, in a general manner, the relations which exist between the pressures which it sustains and the changes of length of its infinitely small dimensions, to obtain the differential equations of the problem under consideration.

26. Equations (3) can be written,

$$\left.\begin{aligned} P &= (H-K)\varepsilon + K(\varepsilon+\varepsilon'+\varepsilon''), \\ Q &= (H-K)\varepsilon' + K(\varepsilon+\varepsilon'+\varepsilon''), \\ R &= (H-K)\varepsilon'' + K(\varepsilon+\varepsilon'+\varepsilon''). \end{aligned}\right\} \tag{5}$$

Calling θ the relative variation of the volume, or cubic dilatation, we may, because of the small values of the deformations, write

$$\theta = \varepsilon + \varepsilon' + \varepsilon''. \tag{6}$$

Placing $H - K = 2\mu$ and $K = \lambda$, we have

$$\left.\begin{aligned} P &= \lambda\theta + 2\mu\varepsilon, \\ Q &= \lambda\theta + 2\mu\varepsilon', \\ R &= \lambda\theta + 2\mu\varepsilon''. \end{aligned}\right\} \tag{7}$$

Each of the tractions or pressures is then the sum of a term proportional to the cubic dilatation and of a term proportional to the linear dilatation parallel to the pressure considered.

27. A liquid parallelopipedon can be in equilibrium only when the pressures exerted on its six faces are equal; and we know besides that the increase of density or negative increase of volume of the liquid is proportional to the pressure. We will then have

$$P = Q = R = \lambda\theta. \tag{8}$$

The same general theory thus comprises both liquids and solids, in admitting the coefficient 2μ of the former to be zero. The variation of this coefficient from zero marks the departure of the body from the perfect liquid state and its approach to that of the solid.

28. *Analytical expression of the elastic forces developed in the motion of a system of molecules, solicited by the forces of attraction or repulsion, and subjected to small displacements from their positions of equilibrium.*

Let x, y, z, and $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, be the rectangular coordinates of the two molecules of the system, whose masses are respectively m and μ, and whose distance apart is r. The intensity of the reciprocal action of the molecules, being exerted along the right line joining them, is

$$m\mu f(r),$$

$f(r)$ being an undetermined function of the distance. If the system is in equilibrium, we have the relations,

$$\left.\begin{aligned} m\,\Sigma\mu f(r)\frac{\Delta x}{r} &= 0, \\ m\,\Sigma\mu f(r)\frac{\Delta y}{r} &= 0, \\ m\,\Sigma\mu f(r)\frac{\Delta z}{r} &= 0. \end{aligned}\right\} \tag{9}$$

At a certain instant, let us suppose that the molecules of the system are displaced from their positions of equilibrium by a very small distance, and let ξ, η, ζ, be the projections of the displacement ε of the molecule m on the axes; let $\xi+\Delta\xi$, $\eta+\Delta\eta$, $\zeta+\Delta\zeta$, be the projections of the displacement of the molecule μ on the axes; and $r+\rho$ the new distance between the molecules. Representing the components of the elastic force parallel to the axes exerted upon the molecule m by all the molecules μ within the

sphere of molecular activity, by $X\varepsilon$, $Y\varepsilon$, $Z\varepsilon$, so that X, Y, Z, are the components of the elastic force for a displacement *unity* in the same directions, we have

$$\left.\begin{aligned} X\varepsilon &= m\,\Sigma\mu\, f(r+\rho)\,\frac{\Delta x + \Delta\xi}{r+\rho}, \\ Y\varepsilon &= m\,\Sigma\mu\, f(r+\rho)\,\frac{\Delta y + \Delta\eta}{r+\rho}, \\ Z\varepsilon &= m\,\Sigma\mu\, f(r+\rho)\,\frac{\Delta z + \Delta\zeta}{r+\rho}. \end{aligned}\right\} \tag{10}$$

29. Developing $f(r+\rho)$, and neglecting the terms of a higher order than those containing ρ, since the displacements are regarded as very small, we obtain, recollecting that $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, are of the same order of magnitude as ρ, while Δx, Δy, Δz, may be of any order whatever,

$$\left.\begin{aligned} X\varepsilon &= m\,\Sigma\mu\,[f(r) + \rho f'(r)]\left(\frac{\Delta x + \Delta\xi}{r}\right)\left(1 - \frac{\rho}{r}\right), \\ &= m\,\Sigma\mu\left[f(r)\frac{\Delta\xi}{r} + \rho f'(r)\frac{\Delta x}{r} - \rho f(r)\frac{\Delta x}{r^2}\right], \\ &= m\,\Sigma\mu\left\{f(r)\frac{\Delta\xi}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\rho\,\frac{\Delta x}{r}\right\}; \end{aligned}\right\} \tag{11}$$

we also have

$$r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2, \tag{12}$$

$$(r+\rho)^2 = (\Delta x + \Delta\xi)^2 + (\Delta y + \Delta\eta)^2 + (\Delta z + \Delta\zeta)^2; \tag{13}$$

from which

$$\rho = \frac{\Delta x\,\Delta\xi + \Delta y\,\Delta\eta + \Delta z\,\Delta\zeta}{r}. \tag{14}$$

Substituting this value of ρ in equations (11), we obtain

$$\left.\begin{aligned} X\varepsilon = m\,\Sigma\mu\Bigg\{&\left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x^2}{r^2}\right)\Delta\xi \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta y}{r^2}\,\Delta\eta \\ &+ \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta z}{r^2}\,\Delta\zeta\Bigg\}. \end{aligned}\right\} \tag{15}$$

Similarly, for the axes Y and Z we get,

$$\left.\begin{aligned} Y\varepsilon = m\,\Sigma\mu\Bigg\{\left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta y}{r^2}\,\Delta\xi \\ + \left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y^2}{r^2}\right)\Delta\eta \\ + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y\,\Delta z}{r^2}\,\Delta\zeta\Bigg\}, \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} Z\varepsilon = m\,\Sigma\mu\Bigg\{\left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta x\,\Delta z}{r^2}\,\Delta\xi \\ + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta y\,\Delta z}{r^2}\,\Delta\eta \\ + \left(\frac{f(r)}{r} + \left[f'(r) - \frac{f(r)}{r}\right]\frac{\Delta z^2}{r^2}\right)\Delta\zeta\Bigg\}. \end{aligned}\right\} \quad (17)$$

Putting $\phi(r)$ for $\frac{f(r)}{r}$ $\psi(r)$ for $f'(r) - \frac{f(r)}{r}$; and $m\frac{d^2\xi}{dt^2}$, $m\frac{d^2\eta}{dt^2}$, $m\frac{d^2\zeta}{dt^2}$, for their equals $X\varepsilon$, $Y\varepsilon$, $Z\varepsilon$, we have

$$\left.\begin{aligned} X\varepsilon = m\frac{d^2\xi}{dt^2} = m\,\Sigma\mu\Bigg\{\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\Delta\xi \\ + \psi(r)\frac{\Delta x\,\Delta y}{r^2}\Delta\eta + \psi(r)\frac{\Delta x\,\Delta z}{r^2}\Delta\zeta\Bigg\}, \\ Y\varepsilon = m\frac{d^2\eta}{dt^2} = m\,\Sigma\mu\Bigg\{\psi(r)\frac{\Delta x\,\Delta y}{r^2}\Delta\xi \\ + \left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta\eta + \psi(r)\frac{\Delta y\,\Delta z}{r^2}\Delta\zeta\Bigg\}, \\ Z\varepsilon = m\frac{d^2\zeta}{dt^2} = m\,\Sigma\mu\Bigg\{\psi(r)\frac{\Delta x\,\Delta z}{r^2}\Delta\xi \\ + \psi(r)\frac{\Delta y\,\Delta z}{r^2}\Delta\eta + \left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\Delta\zeta\Bigg\}; \end{aligned}\right\} \quad (18)$$

which give the values of the component elastic forces developed in any molecule of the medium, when the displacements are small.

30. If the displacement is only in the direction of each axis in succession, we have the following groups of equations.

Of x:
$$\left.\begin{aligned} X_1 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\Delta\xi, \\ Y_1 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right]\Delta\xi, \\ Z_1 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right]\Delta\xi\,; \end{aligned}\right\} \tag{19}$$

Of y:
$$\left.\begin{aligned} X_2 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right]\Delta\eta, \\ Y_2 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta\eta, \\ Z_2 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right]\Delta\eta\,; \end{aligned}\right\} \tag{20}$$

Of z:
$$\left.\begin{aligned} X_3 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right]\Delta\zeta, \\ Y_3 &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right]\Delta\zeta, \\ Z_3 &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right]\Delta\zeta. \end{aligned}\right\} \tag{21}$$

31. Combining the above equations, we have

$$\left.\begin{aligned} X\varepsilon &= X_1 + X_2 + X_3, \\ Y\varepsilon &= Y_1 + Y_2 + Y_3, \\ Z\varepsilon &= Z_1 + Z_2 + Z_3. \end{aligned}\right\} \tag{22}$$

From Eqs. (19) we see that the total intensity $\sqrt{X_1^2 + Y_1^2 + Z_1^2}$ of the elastic force developed is proportional to the relative displacement $\Delta\xi$, and since the axis has been assumed arbitrarily, it can be said, in general, that the total intensity, $\varepsilon\sqrt{X^2 + Y^2 + Z^2}$, developed, is directly proportional to the general relative displacement,

$$\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2} = \varepsilon.$$

From Eqs. (22) we conclude that the component intensity of the elastic force developed in the direction of any axis, due to any displacement, is equal to the sum of the three component intensities developed by three successive displacements along these axes, equal to the respective projections of the general displacement on these axes.

32. Of the nine coefficients of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, given in Eqs. (18), six only are distinct. Representing these by

$$\left.\begin{aligned}
A &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right],\\
B &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right],\\
C &= m\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right],\\
D &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta y\,\Delta z}{r^2}\right],\\
E &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta y}{r^2}\right],\\
F &= m\,\Sigma\mu\left[\psi(r)\frac{\Delta x\,\Delta z}{r^2}\right],
\end{aligned}\right\} \qquad (23)$$

we can write Eqs. (22),

$$\left.\begin{aligned}
X\varepsilon &= A\,\Delta\xi + E\,\Delta\eta + F\,\Delta\zeta,\\
Y\varepsilon &= E\,\Delta\xi + B\,\Delta\eta + D\,\Delta\zeta,\\
Z\varepsilon &= F\,\Delta\xi + D\,\Delta\eta + C\,\Delta\zeta;
\end{aligned}\right\} \qquad (24)$$

from which we conclude that the component elastic force developed along any axis, x, for example, by a displacement ε along any other axis y is equal to the component elastic force developed along the axis y by an equal displacement along the axis x.

33. From Eqs. (19–21) we see that when a displacement is made in any direction, the resulting elastic force is not, in general, in the same direction. To find whether we can refer the system to rectangular co-ordinate axes, so that when a displacement is made along such an axis, exceptional elastic forces will be developed, whose total resultant will be in the direction of the displacement, let α, β, γ,

be the angles which the direction of the displacement makes with the axes; λ, μ, and ν, the angles which the resultant elastic force makes with the same axes; then we have

$$\left.\begin{aligned}
\cos\lambda &= \frac{X}{\sqrt{X^2+Y^2+Z^2}},\\
\cos\mu &= \frac{Y}{\sqrt{X^2+Y^2+Z^2}},\\
\cos\nu &= \frac{Z}{\sqrt{X^2+Y^2+Z^2}},\\
\cos\alpha &= \frac{\Delta\xi}{\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},\\
\cos\beta &= \frac{\Delta\eta}{\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},\\
\cos\gamma &= \frac{\Delta\zeta}{\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},
\end{aligned}\right\} \tag{25}$$

Since the resultant intensity of the elastic force is proportional to the displacement, we may let K represent the intensity of the elastic force corresponding to a displacement equal to unity. K varying with the direction of the displacement, we can then place $K\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}$ for its representative, $\varepsilon\sqrt{X^2+Y^2+Z^2}$, and the first of Eqs. (25) will become

$$\left.\begin{aligned}
\cos\lambda &= \frac{X\varepsilon}{K\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},\\
\cos\mu &= \frac{Y\varepsilon}{K\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},\\
\cos\nu &= \frac{Z\varepsilon}{K\sqrt{\Delta\xi^2+\Delta\eta^2+\Delta\zeta^2}},
\end{aligned}\right\} \tag{26}$$

Substituting the values of X, Y, Z, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, derived from these equations in Eqs. (24), after omitting the common factor ε, we have

$$\left.\begin{aligned}
K\cos\lambda &= A\cos\alpha + E\cos\beta + F\cos\gamma,\\
K\cos\mu &= E\cos\alpha + B\cos\beta + D\cos\gamma,\\
K\cos\nu &= F\cos\alpha + D\cos\beta + C\cos\gamma.
\end{aligned}\right\} \tag{27}$$

Applying the conditions

$$\lambda = \alpha, \qquad \mu = \beta, \qquad \nu = \gamma,$$

we have the equations of condition,

$$\left.\begin{aligned} (A - K)\cos\alpha + E\cos\beta + F\cos\gamma &= 0, \\ E\cos\alpha + (B - K)\cos\beta + D\cos\gamma &= 0, \\ F\cos\alpha + D\cos\beta + (C - K)\cos\gamma &= 0; \end{aligned}\right\} \qquad (28)$$

together with $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \qquad (29)$

which make four equations containing the four unknown quantities α, β, γ, and K.

34. In order that Eqs. (28) may be true for the same set of values of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, we must have the determinant

$$\left\{\begin{matrix} A - K, & E, & F, \\ E, & B - K, & D, \\ F, & D, & C - K, \end{matrix}\right\} = 0. \qquad (30)$$

Multiplying Eqs. (28) respectively by D, F, and E, we get

$$\left.\begin{aligned} (AD - KD)\cos\alpha + DE\cos\beta + DF\cos\gamma &= 0, \\ EF\cos\alpha + (BF - FK)\cos\beta + DF\cos\gamma &= 0, \\ EF\cos\alpha + DE\cos\beta + (CE - EK)\cos\gamma &= 0, \end{aligned}\right\} \qquad (31)$$

Placing

$$\left.\begin{aligned} AD - EF &= aD, \\ BF - DE &= a'F, \\ CE - DF &= a''E, \end{aligned}\right\} \qquad (32)$$

we have

$$\left.\begin{aligned} (a-K)D\cos\alpha + EF\cos\alpha + DE\cos\beta + DF\cos\gamma &= 0, \\ (a'-K)F\cos\beta + EF\cos\alpha + DE\cos\beta + DF\cos\gamma &= 0, \\ (a''-K)E\cos\gamma + EF\cos\alpha + DE\cos\beta + DF\cos\gamma &= 0; \end{aligned}\right\} \qquad (33)$$

from which

$$\left.\begin{aligned} (a - K)D\cos\alpha &= (a' - K)F\cos\beta \\ &= (a'' - K)E\cos\gamma = P; \end{aligned}\right\} \qquad (34)$$

whence,

$$\left.\begin{aligned} \cos\alpha &= \frac{P}{(a - K)D}, \\ \cos\beta &= \frac{P}{(a' - K)F}, \\ \cos\gamma &= \frac{P}{(a'' - K)E}. \end{aligned}\right\} \qquad (35)$$

Substituting these values in the first of Eqs. 33, we obtain

$$1 + \frac{EF}{(a-K)D} + \frac{DE}{(a'-K)F} + \frac{DF}{(a''-K)E} = 0. \quad (36)$$

Clearing of fractions, we have

$$\left.\begin{array}{r} DEF(K-a)(K-a')(K-a'') - E^2F^2(K-a')(K-a'') \\ -D^2E^2(K-a'')(K-a) \\ -D^2F^2(K-a)(K-a') \end{array}\right\} = 0. \quad (37)$$

If DEF be positive, supposing

1°, that $a < a' < a''$, by substituting for K, in succession in Eq. (37), $-\infty$, a, a', a'', $+\infty$, we obtain

$$\begin{array}{l} -\infty, \\ -E^2F^2(a'-a)(a''-a), \\ +D^2E^2(a''-a')(a'-a), \\ -D^2F^2(a''-a)(a''-a'), \\ +\infty, \end{array}$$

which, since there are three variations in the signs, shows that Eq. (37) has three real roots, one lying between a and a', one between a' and a'', and the third between a'' and ∞. Similarly, if DEF be negative, the real roots will be found as above.

2°. If two of the quantities a, a', a'', are equal, as, for example, $a' = a''$, Eq. (37) reduces to

$$\left.\begin{array}{l} (K-a')\{[EF(K-a')][D(K-a)-EF] \\ \qquad -D^2(F^2+E^2)(K-a)\} \end{array}\right\} = 0. \quad (38)$$

which gives a real root between a and a', a second equal to a', and a third greater than a'.

3°. If the three quantities a, a', a'', are equal, Eq. (37) reduces to

$$(K-a)^2[DEF(K-a) - E^2F^2 - D^2E^2 - D^2F^2] = 0, \quad (39)$$

giving two real roots, each equal to a, and one greater than a. Each of these real roots of K, being substituted in one of Eqs. (35), will enable us to find values for each of the cosines between $+1$ and -1, and hence a given direction for each value of K, or in all *three directions.*

35. We therefore conclude, *that the total elastic force developed by any displacement is not in general in the line of direction of the displacement, but oblique to it; that there are three directions at right angles to each other, and, in general, only three, along which, if the displacement be made, the resultant elastic force developed will be in the direction of the displacement.*

36. These three directions are called *principal axes.* They are not specific lines in a body, but simply mark directions along which the above property exists.

37. The angle which the direction of the displacement and the resultant elastic force make with each other is given by

$$\left.\begin{aligned}\cos U &= \frac{X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta}{\sqrt{X^2 + Y^2 + Z^2}\,\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}} \\ &= \frac{X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta}{K\sqrt{\Delta\xi^2 + \Delta\eta^2 + \Delta\zeta^2}};\end{aligned}\right\} \quad (40)$$

and, if the displacement be equal to unity, we have

$$\left.\begin{aligned}K\cos U &= X\,\Delta\xi + Y\,\Delta\eta + Z\,\Delta\zeta \\ &= X\cos\alpha + Y\cos\beta + Z\cos\gamma.\end{aligned}\right\} \quad (41)$$

38. *Surfaces of Elasticity.* If now distances which are proportional to the elastic forces developed by a constant displacement, equal to unity, for example, in each direction, be laid off in all directions from any point of the medium, the extremities of these lines will form a surface which may be called a *surface of elasticity.* But, as for each direction there are two things to consider, viz., the intensity of the elastic force and the angle which its direction makes with the displacement, we cannot, in general, construct a surface which would unite these two particulars.

39. It will be shown, hereafter, upon what grounds we can disregard, in optics, that component of the elastic force, $K \sin U$, which is perpendicular to the displacement, and consider, as alone effective, the component whose intensity is represented by $K \cos U$, parallel to the displacement.

40. Assuming then, for the present, that the effective elastic force caused by a displacement equal to unity is given by Eq. (41), and substituting the radius vector r for the first member, and the

values of X, Y, Z, from Eqs. (24), and for $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, their values for a displacement unity, we get

$$\left.\begin{aligned} r = A\cos^2\alpha + 2E\cos\alpha\cos\beta + 2F\cos\alpha\cos\gamma \\ + B\cos^2\beta + 2D\cos\beta\cos\gamma + C\cos^2\gamma, \end{aligned}\right\} \quad (42)$$

the polar equation of a surface of elasticity of the medium.

Substituting for $\cos\alpha$, $\cos\beta$, $\cos\gamma$, their values $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, and for r its equal $\sqrt{x^2+y^2+z^2}$, Eq. (42) becomes

$$\sqrt{x^2+y^2+z^2} = \frac{1}{x^2+y^2+z^2}[Ax^2+By^2+Cz^2+2Exy+2Fxz+2Dyz]. \quad (43)$$

41. Assuming that the radius vector is proportional to the square root of the elastic force, the equation takes the form

$$(x^2+y^2+z^2)^2 = Ax^2+By^2+Cz^2+2Exy+2Fxz+2Dyz, \quad (44)$$

which is the equation of *Fresnel's Surface of Elasticity.*

42. By assuming each radius vector proportional to the reciprocal of the square root of the elastic force, Eq. (42) becomes

$$1 = Ax^2 + By^2 + Cz^2 + 2Exy + 2Fxz + 2Dyz, \quad (45)$$

which is the equation of what has been designated as the *inverse ellipsoid of elasticity*, or the *first ellipsoid*, and is called the ellipsoid E.

43. ***Surfaces of Elasticity referred to Principal Axes.*** Principal axes are those along which, if the displacement be made, the resultant elastic forces developed will be wholly in the same direction. We have seen that, in any homogeneous medium, there are in general three, and only three, such directions. Making $\Delta\eta$, $\Delta\zeta$; $\Delta\xi$, $\Delta\zeta$; $\Delta\xi$, $\Delta\eta$, respectively equal to zero in Eqs. (24), and placing A, B, C, equal to a^2, b^2, c^2, respectively, we have

$$\left.\begin{aligned} X = a^2\Delta\xi, \quad Y = b^2\Delta\eta, \quad Z = c^2\Delta\zeta, \\ E = F = D = 0, \end{aligned}\right\} \quad (46)$$

and Eq. (44) reduces to

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2; \quad (47)$$

and Eq. (45) to $\quad a^2x^2 + b^2y^2 + c^2z^2 = 1. \quad (48)$

Fresnel's surface of elasticity, Eq. (47), is of the fourth order, its equation being of the fourth degree. Figure (3) represents one-quarter of the principal section made by the plane ac, turned about the axis b through an angle of 90°. Taking the axes to be

$$a = 1.53, \qquad b = 1.32, \qquad c = 1.00,$$

we may, by Eq. (47), readily construct the principal sections. Thus, since

$$r^4 = a^2x^2 + b^2y^2 + c^2z^2, \qquad \therefore\ a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma = r^2;$$

we have for the intersection by the plane ac, $\beta = 90°$, and

$$r^2 = a^2 \cos^2 \alpha + c^2 \cos^2 \gamma = r'^2 + r''^2,$$

when $\qquad r' = a \cos \alpha, \quad$ and $\quad r'' = c \cos \gamma = c \sin \alpha.$

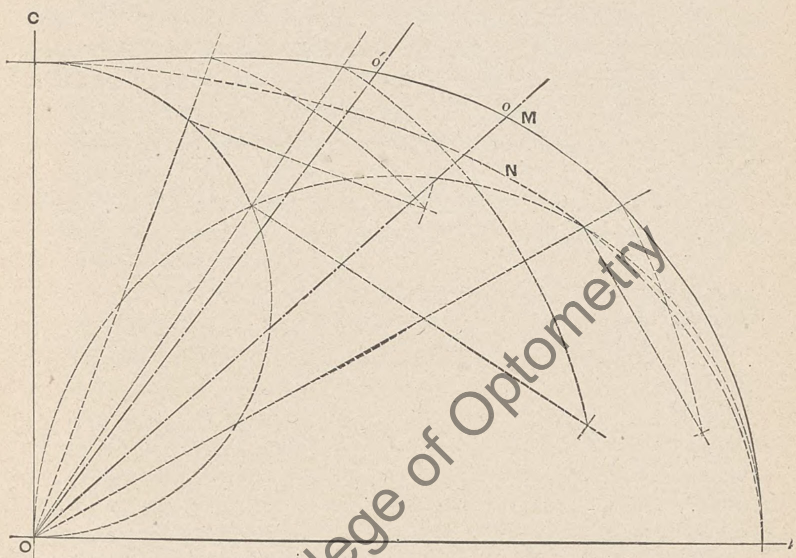

Figure 3.

Therefore r is equal to the hypothenuse of the right-angled triangle on r' and r''; hence, describe semicircles on a and c; draw any right line from O, and lay off on it a distance equal to the hypothenuse on the intercepts of the two circles, and this will be a point of the curve. Three such points are constructed in the

figure. The curve CMA is the intersection of Fresnel's surface with *ac*; the curve CNA is that of the ellipsoid whose semi-axes coincide with and are equal to those of the surface of Fresnel; O*o* and O*o*′ are the traces of the cyclic planes which contain the axis *b* of the surface of elasticity and of the ellipsoid respectively. The principal elasticities in crystals never differ so much as those assumed above, and therefore, in many cases, the departure of the surface from the ellipsoid is negligible.

WAVES.

44. The elastic forces of the medium, developed by the assumed arbitrary displacement of a molecule, will propagate the motion in all directions from the point of initial disturbance. As an ever-enlarging volume becomes involved in this disturbance, each molecule takes up a motion exactly similar to that of its predecessor, which it transmits in turn to the next molecule. This transfer is complete when a single pulse traverses the medium, and is both complete and continuous when these pulses are successively continuous.

In this latter case the exciting cause acts for a definite portion of time. Representing by a series of dots, *a*. . . . *b*, the position of

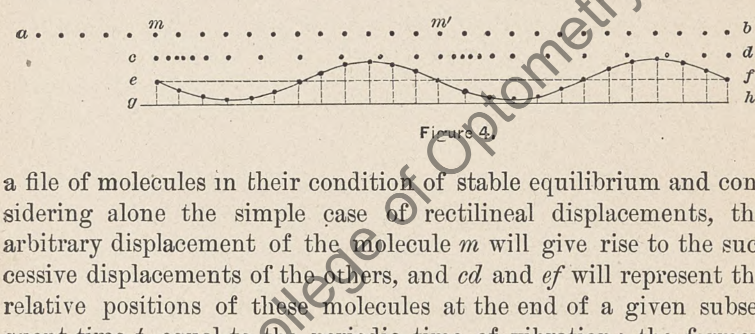

Figure 4.

a file of molecules in their condition of stable equilibrium and considering alone the simple case of rectilineal displacements, the arbitrary displacement of the molecule *m* will give rise to the successive displacements of the others, and *cd* and *ef* will represent the relative positions of these molecules at the end of a given subsequent time *t*, equal to the periodic time of vibration; the former, when the displacements are parallel to the direction of disturbance propagation, and the latter, when at right angles to this direction.

While, therefore, any molecule *m* is describing its orbit, the disturbance is being propagated in all directions, and, at the instant the orbit of *m* is completed, the disturbance will have reached

another molecule m', on the same line of direction, which will then, for the first time, begin to move; and the molecules m and m' will, thereafter, always be at the same relative distances from their origins.

45. While this undulatory motion is being propagated, molecules will be found between m and m', with all degrees of displacement, both as to amount and direction of motion, consistent with the dimensions and shapes of their orbits. If the velocity of wave propagation be constant in all directions, the form assumed by the bounding surface containing the disturbed molecules will be spherical; but if the velocity vary, the form will depend upon the law of its variation.

46. This continuous transmission in any given direction of a *relative* state of the molecules, while the motion of each molecule is orbital, is characteristic of an *undulation*.

47. The term *phase* is used to express the condition of a molecule with respect to its *displacement* and the *direction* of its *motion*. Molecules are said to be in *similar* phases, when moving in parallel orbital elements and in the same direction; and in *opposite* phases, when moving in parallel orbital elements and in opposite directions. More generally, similar phases are those in which the anomalies of the molecule are the same, and opposite phases those in which the anomalies differ by 180°. (By *anomaly* is meant the angular distance from an assumed right line.)

48. A *wave* is the particular *form of aggregation* assumed by the molecules between the nearest two consecutive surfaces in which similar phases simultaneously exist throughout.

A *wave front* is that surface which contains molecules only in the same phase; it is generally understood to refer to the surface upon which the molecules are just beginning to move. The velocity of a wave front will always be that of the disturbance propagation.

A *wave length* is the interval, measured in the direction of wave propagation, between the nearest two consecutive surfaces upon which the molecules have similar phases.

The *amplitude* of the undulation is the maximum displacement of the molecule from its place of rest.

49. From a consideration of the nature of an undulation, we see at once that, if λ be the wave length, τ the periodic time, and V the velocity of wave propagation, we will have

$$V = \frac{\lambda}{\tau}, \tag{49}$$

and the values of V, λ, and τ are each, theoretically, independent of the amplitude.

50. To find an expression for the displacement of a molecule at any time during the transmission of an undulation, let x be the distance of the molecule from the origin of disturbance, t the time from the epoch, τ the periodic time of the molecule, λ the wave length, and V the velocity of wave propagation. Now, whatever be the displacement δ of the molecule x, at the time t, an equal displacement (neglecting the loss due to increased distance from the origin) will exist for another molecule at a distance $x + Vt'$, at the time $t + t'$. This condition gives, whatever be the value of t',

$$\delta = \phi(x, t) = \phi(x + Vt', t + t'). \tag{50}$$

$x + Vt'$ is the distance from the origin to the wave front at a time t subsequent to the instant at which it was at x. Hence the molecule x is behind the wave front a distance $Vt - x$, and the displacement, $\phi(x, t)$, may be replaced by $\phi(Vt - x)$; therefore we have

$$\delta = \phi(x, t) = \phi(Vt - x), \tag{51}$$

as the form of the function.

51. We have implicitly assumed the medium to be in a state of stable equilibrium during the passage of the undulation, and, therefore, the molecule will necessarily describe a closed orbit about its place of relative rest. This orbit may, from the circumstances of the case, be of the most varied character, and, after the energy due to the disturbance has been dissipated, the molecule will resume its original place of relative rest, until again displaced by some new disturbance. It is necessary, in this discussion, to consider those disturbances alone which are regular and periodic, and to consider the orbit after it has become determinate. We therefore limit the discussion to that of the regular periodic disturbance, and the orbit to that of the ellipse or any of its particular cases, such as the ellipse, the circle, or the right line.

52. ***Simple Harmonic Motion.*** If a point a (Fig. 5) move uniformly in a circular orbit, the distance of its projection

From the centre, upon the vertical diameter, can always be found from the equation

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right), \tag{52}$$

in which y is the required displacement at the time t, a is the amplitude or maximum displacement, τ the periodic time, and α the angle included between the horizontal diameter and that passing through the origin of motion.

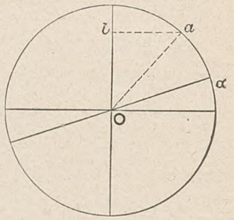

Figure 5.

The angle $\frac{2\pi t}{\tau} + \alpha$ is called the phase of the vibration, and may be made of any value by changing the arbitrary arc α, the time t, or both together. The same value will apply to motion along any diameter. Such motions are called simple harmonic motions. It may easily be shown that any two simple harmonic motions, in one line and of the same period, may be compounded into a single simple harmonic motion of the same period, but whose amplitude is equal to the diagonal of a parallelogram constructed on the amplitudes of the components inclined to each other by an angle equal to their difference of phase.

53. ***The Harmonic Curve.*** If the motion of a point be compounded of a rectilineal harmonic vibration, and of uniform motion in a straight line perpendicular to the vibration, the point will describe a plane curve, which is called the *harmonic curve.*

Let the vibration be along the axis of y, and uniform motion along the axis x; we will then have

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right) \tag{53}$$

for the ordinates, and

$$x = Vt \tag{54}$$

for the abscissas, due to the uniform motion. Combining these equations, eliminating t, and replacing $V\tau$ by its equal λ, Eq. (49), page 40, we have, for the equation of the harmonic curve,

$$y = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right); \tag{55}$$

in which λ is the wave length. Substituting for x, $x \pm i\lambda$, the value of y remains the same for all integral values of i. The curve, therefore, consists of an infinite number of similar parts, which are symmetrical with respect to the axis of x.

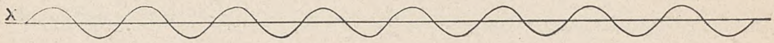

Figure 6.

54. To construct the curve by points, divide the circumference into any number, as twelve, equal parts; lay off on the axis of abscissas twelve equal distances, corresponding to the positions of the point in uniform motion, erect ordinates at these points and make them equal to the corresponding displacements at the given times, and we have the curve as follows:

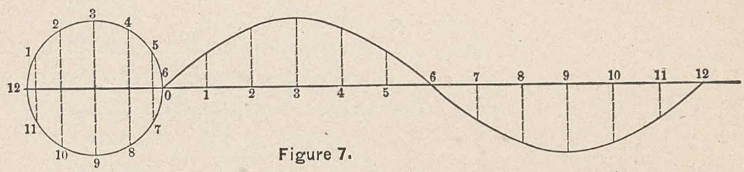

Figure 7.

55. The varying velocities of a point of a simple pendulum in motion can be represented by the ordinates of the harmonic curve; and because of this analogy all vibrations represented by these curves are called simple or pendular vibrations. The vibration is taken to be the complete oscillation, from the time at which the moving point was in one position until it returns to the same position again. By this definition, the duration of the vibration of a second's pendulum would be *two* seconds, and not *one* second.

56. ***Composition of Harmonic Curves.*** Let

$$y' = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right), \tag{56}$$

$$y'' = b \sin\left(\frac{2\pi x}{\lambda} + \beta\right), \tag{57}$$

be the equations of any two harmonic curves, having the same wave length, but different amplitudes. The resultant value of y will be

$$y = c \sin\left(\frac{2\pi x}{\lambda} + \gamma\right), \tag{58}$$

which is the equation of another harmonic curve, of equal wave length, but of different amplitude from either of the components.

The values of c and γ are given by

$$c \cos \gamma = a \cos \alpha + b \cos \beta, \tag{59}$$

$$c \sin \gamma = a \sin \alpha + b \sin \beta, \tag{60}$$

$$c = \sqrt{a^2 + b^2 + 2ab \cos(\alpha - \beta)}. \tag{61}$$

From the last equation we see that c may have any value between the sum and difference of a and b, depending upon the value of the difference of phase, $\alpha - \beta$, of the components.

By a similar process, it can be shown that any number of component harmonic curves, of the same wave length, may be compounded into a single resultant harmonic curve having an equal wave length, but whose amplitude and phase differ in general from those of any of its components.

57. If the component curves have different wave lengths, they cannot be compounded into a single harmonic curve; but when their wave lengths are commensurable, they can be compounded into a periodic curve, whose period is the least common multiple of their several periods. Thus, in the first case, where the wave lengths are unequal and incommensurable for the resultant ordinate,

$$y = a \sin\left(\frac{2\pi x}{\lambda'} + \alpha\right) + b \sin\left(\frac{2\pi x}{\lambda''} + \beta\right) + c \sin\left(\frac{2\pi x}{\lambda'''} + \gamma\right) + \ldots., \tag{62}$$

in which the period is infinite, or the curve is non-periodic.

In the second case, let

$$\lambda' = \frac{\lambda}{m}, \qquad \lambda'' = \frac{\lambda}{n}, \qquad \lambda''' = \frac{\lambda}{r}, \ \ldots\ldots.., \tag{63}$$

m, n, r, being integers; then the above equation becomes

$$y = a \sin\left(\frac{2\pi m x}{\lambda} + \alpha\right) + b \sin\left(\frac{2\pi n x}{\lambda} + \beta\right) + c \sin\left(\frac{2\pi r x}{\lambda} + \gamma\right) + \ldots., \tag{64}$$

which, although not admitting of reduction to a simpler form, gives

constantly recurring values of y when for x we substitute $x + \lambda$. The wave length of the resultant curve is therefore λ, and the curve is periodic.

58. The *forms* of the component curves depend only upon the wave lengths and amplitudes; but their positions on the axis depend on the values of the phase α, β, γ, etc. By assigning arbitrary values to these, we may shift any curve along the axis any desired part of its wave length. Any such shifting for any one or more of the component curves will necessarily alter the form of the resultant curve, but will not change its wave length.

59. If the wave length of the resultant curve be assumed, the wave lengths of its components may be all possible aliquot parts of λ, and the number of the possible components is therefore unlimited. Therefore every possible curve of wave length λ, which could be so constructed from such component curves, would be found among those produced by placing, along the same axis, an unlimited number of harmonic curves, as components, with wave lengths λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, etc., . . .

By varying the amplitudes of the components and shifting them arbitrarily along the axis, an infinite number of resultants can be produced, all having the same wave length λ. Fourier's theorem demonstrates that every possible variety of periodic curve, of given wave length λ, can be so produced, provided that the ordinate is always finite and that the moving point is assumed to move always in the same direction.

60. A periodic series is one whose terms contain sines or cosines of the variable, or of its multiples; thus,

$$A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \ldots . A_n \cos nx + \ldots .$$

is a periodic series. This series goes through a succession of values as the arc increases from 0 to 2π; for, every term has the same value at the end and at the beginning of that period, and this continuously, so that whatever n may be, the period of the function is 2π.

61. Fourier's Theorem has for its object the determination of the unknown constants, $A_0, A_1, A_2, \ldots B_1, B_2, B_3, \ldots$, and the determination of the conditions by which any given function, $y = f(x)$, can be expressed in the form of

$$\left.\begin{aligned} y = f(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \ldots . \\ + B_1 \sin x + B_2 \sin 2x + \text{etc.} \ldots . \end{aligned}\right\} \quad (65)$$

The non-periodic term A_0 is introduced to make the theorem conform to the most general case. If the function is capable of expression in periodic terms only, then $A_0 = 0$; this fact can only be determined by considering each special case.

The equation which expresses the mathematical statement of Fourier's Theorem is

$$y = y_0 + \Sigma_{i=1}^{i=\infty} C_i \sin\left(\frac{2i\pi t}{\tau} + \alpha_i\right), \quad (66)$$

in which y_0 is the mean value of y, and each of the variable terms represents, by itself, a harmonic vibration of which the period is an aliquot part of the whole period τ.

62. ***Wave Function.*** Resuming Eq. (51),

$$\delta = \phi(x, t) = \varphi(Vt - x),$$

we see that, since the displacement δ passes through all of its values while the undulation advances a distance equal to its wave length λ, it has the properties of simple harmonic motion, and, therefore, may be written

$$\delta = a \sin \frac{2\pi}{\lambda}(Vt - x). \quad (67)$$

This is called the wave function. By making t vary continuously through all values from $t = \frac{x}{V}$ to $t = \frac{x+\lambda}{V}$, δ will increase from zero to $+a$, decrease then to $-a$, and finally return to zero, during the time $\frac{\lambda}{V}$, which is evidently the interval of time required for the undulation to pass over the wave length λ. Again, supposing t to remain constant and x to vary through all values from $Vt - \lambda$ to Vt, we obtain again all possible values of the displacement, which values will evidently belong, at the same instant, to all molecules in the wave length. The following diagram illustrates the two cases:

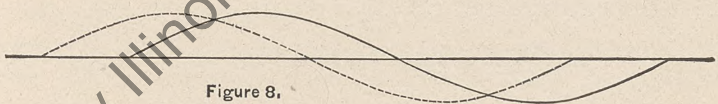

Figure 8.

By the addition of an arbitrary arc we can cause the displacement to take any one of its values, at any time t, and thus change our origin at pleasure.

63. The corresponding expression for the velocity of the molecule in its rectilineal orbit, sometimes called the *velocity* of the *wave element*, in contradistinction to the *velocity* of *wave propagation*, is given by

$$u = a \cos \frac{2\pi}{\lambda} (Vt - x). \tag{68}$$

64. The principle of the coexistence and superposition of small motions is shown in Mechanics to be applicable to planetary perturbations. It is, for similar reasons, applicable to the determination of the resultant displacement of a single molecule, arising from the concurrent effect of many disturbing causes acting separately. The acceptance of this principle is equivalent to assuming that the several displacements are so small that their products and powers higher than the first are negligible with respect to the displacements themselves; and it embodies the primary supposition that the intensity of elastic forces developed varies directly with the degree of displacement.

65. ***Wave Interference.*** If we apply this principle to determine the displacement of a molecule by two disturbing causes, giving rise to two undulations of the same wave length, we will have for the first,

$$\delta' = a' \sin \left[\frac{2\pi}{\lambda} (Vt - x) + A' \right]; \tag{69}$$

for the second,

$$\delta'' = a'' \sin \left[\frac{2\pi}{\lambda} (Vt - x) + A'' \right]. \tag{70}$$

The total displacement will be

$$\left.\begin{aligned} \delta' + \delta'' = \delta = (a' \sin A' + a'' \sin A'') \cos \left[\frac{2\pi}{\lambda} (Vt - x) \right] \\ + (a' \cos A' + a'' \cos A'') \sin \left[\frac{2\pi}{\lambda} (Vt - x) \right], \end{aligned}\right\} \tag{71}$$

which may be put under the form

$$\delta = a \sin \left[\frac{2\pi}{\lambda} (Vt - x) + A \right], \tag{72}$$

by placing
$$\left.\begin{aligned} a \cos A &= a' \cos A' + a'' \cos A'', \\ a \sin A &= a' \sin A' + a'' \sin A''. \end{aligned}\right\} \quad (73)$$

Whence,
$$a^2 = a'^2 + a''^2 + 2a'a'' \cos (A' - A''), \quad (74)$$

$$\tan A = \frac{a' \sin A' + a'' \sin A''}{a' \cos A' + a'' \cos A''}. \quad (75)$$

By Eq. (72) we see that the resultant undulation is of the same wave length as the components; that the maximum displacement of the resultant undulation is not, in general, equal to that of either of the components, and that it does not occur at the same time nor place with either of them.

66. Taking the square root of Eq. (74), we have

$$a = \sqrt{a'^2 + a''^2 + 2a'a'' \cos (A' - A'')}; \quad (76)$$

from which it is seen that, when $A' - A'' = 0$, $a = a' + a''$; and, when $A' - A'' = 180°$, $a = a' - a''$. Hence, in Eq. (75), $A = A' = A''$ in the first case, and $A = A' = 180° + A''$ in the second. The maximum displacement, then, of the resultant undulation may vary between the sum and difference of the maximum displacements of the two component undulations, depending upon the difference of phase.

If, in the two component undulations, $a' = a''$, a will be equal to $2a'$ when $A' = A''$, and vary from this value to zero as the difference of phase $A' - A''$ passes from zero to 180°.

Substituting, in the expression for the displacement, $A' \pm 180°$ for A', we will have

$$a' \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A' + \pi\right] = a' \sin \left[\frac{2\pi}{\lambda}\left(Vt - x \pm \frac{\lambda}{2}\right) + A'\right], \quad (77)$$

which is exactly the same as

$$a' \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A'\right],$$

when for x we put $x \mp \frac{\lambda}{2}$.

Therefore, if we suppose that two undulations of the same wave length, starting in the same phase, meet after travelling over routes which differ by one-half the wave-length, there will be no displace-

ment of the molecule at the place of meeting, and complete interference will result.

The diagrams of Figure 9 illustrate the composition of two undulations of equal wave length, having the same phase in the first case, and opposite phases in the second and third cases. In AB, the amplitude of the resultant undulation a is equal to the sum of the amplitudes of the component undulations, a' and a''; in A′B′ and A″B″, equal to the difference of the amplitudes. In A″B″, the displacement of the molecules is zero, and the two components mutually destroy each other's action.

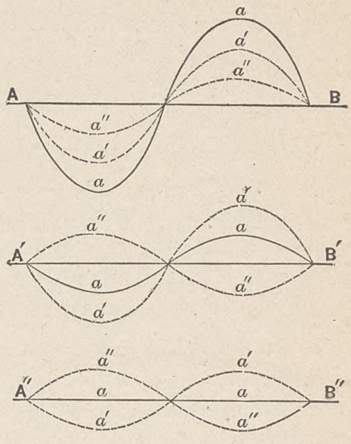

Figure 9.

67. *Interference of any Number of Undulations.*

1° Case. When the component undulations have the same wave length.

Let
$$\left.\begin{aligned}
\delta' &= \alpha' \sin\left[\frac{2\pi}{\lambda}(Vt-x)+A'\right],\\
\delta'' &= \alpha'' \sin\left[\frac{2\pi}{\lambda}(Vt-x)+A''\right],\\
\delta''' &= \alpha''' \sin\left[\frac{2\pi}{\lambda}(Vt-x)+A'''\right],\\
\text{etc.},& \qquad \text{etc.}, \ldots,
\end{aligned}\right\} \qquad (78)$$

be the values of the several component displacements. By addition we have

$$\left.\begin{aligned}
&\delta'+\delta''+\delta'''+\text{etc.}\\
&=(\alpha'\sin A'+\alpha''\sin A''+\alpha'''\sin A'''+\text{etc.})\cos\left[\frac{2\pi}{\lambda}(Vt-x)\right]\\
&+(\alpha'\cos A'+\alpha''\cos A''+\alpha'''\cos A'''+\text{etc.})\sin\left[\frac{2\pi}{\lambda}(Vt-x)\right].
\end{aligned}\right\} (79)$$

The second member may be placed under the form of

$$\left.\begin{aligned} &\alpha \sin A \cos \frac{2\pi}{\lambda}(Vt - x) + \alpha \cos A \sin \frac{2\pi}{\lambda}(Vt - x) \\ &\qquad\qquad = \alpha \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A\right] = \delta. \end{aligned}\right\} \quad (80)$$

From which we conclude that the resultant undulation will have the same wave length as that of the components, but that in general the maximum displacement and the phase at the time t will be different from those of its components.

68. 2° CASE. Component undulations of different wave lengths.

If the wave lengths are different, the displacements are of the form

$$\alpha' \sin \left[\frac{2\pi}{\lambda}(Vt - x) + A'\right],$$

$$\alpha'' \sin \left[\frac{2\pi}{\lambda'}(Vt - x) + A''\right],$$

which cannot be combined into a single circular function of the same form. If in addition the wave velocities also differ, they may be combined if $\frac{V}{\lambda} = \frac{V'}{\lambda'}$. Hence, undulations of different wave lengths cannot destroy each other, and the combined effect of several undulations upon a single molecule will be equal to the algebraic sum of their separate effects. If this sum should reduce to zero for a given molecule, it will differ from zero for the molecules immediately preceding and following it.

69. ***The Principle of Huyghens.*** Since the displacement of any molecule is the cause of the subsequent displacement of other molecules, we may regard the displacement of the molecules upon any wave front as the cause of the subsequent displacement of the molecules upon any other front which the wave afterwards reaches. We may therefore consider each molecule of the wave front in any of its anterior positions as being the origin, and its displacement as the cause of secondary waves, each of which proceed with the same velocity. The aggregate effect of all these

secondary waves upon any other molecule beyond, or its resultant displacement, will evidently be the same as that due to the primary wave itself. This principle is known as that of Huyghens, and, together with the principle of interference, is exceedingly fruitful in explaining many of the phenomena of wave motion in sound and light.

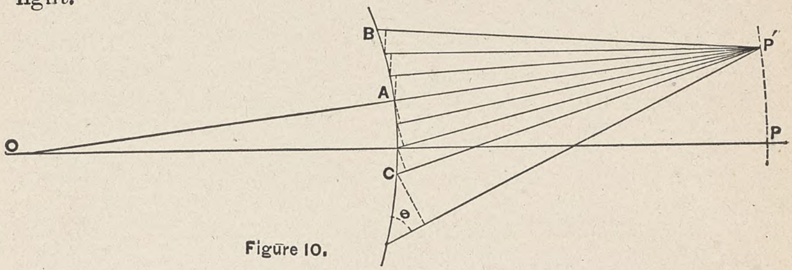

Figure 10.

Let O be the origin of disturbance, and BAC the great wave in any of its anterior positions before reaching a molecule P′; let AP′ $= l'$, AB $=$ AC $= b$; let dz be any indefinitely small part of the wave front, and θ the angle made by the wave front with any right line l drawn from P′ to any point of the wave front, at a distance z from A; then

$$l' = \text{AP}' = \sqrt{(l^2 - 2lz\cos\theta + z^2)} \tag{81}$$

$$= l - z\cos\theta + \frac{z^2}{2l}\sin^2\theta + \text{etc.} \tag{82}$$

$$= l - z\cos\theta, \tag{83}$$

for all points of BAC near to A, and for which $\frac{z^2}{2l}$ is insignificant.

The displacement at P′, due to the secondary waves originating in dz, will therefore be

$$\delta' = \frac{\alpha\, dz}{\text{AP}'}\sin\frac{2\pi}{\lambda}\,(Vt - l + z\cos\theta). \tag{84}$$

Replacing AP′ by l, and integrating, we have for the resultant displacement of P′ due to the great wave,

$$\left.\begin{aligned}\delta = \Sigma\delta' &= \frac{\alpha}{l}\int dz\sin\frac{2\pi}{\lambda}\,(Vt - l + z\cos\theta)\\ &= -\frac{\alpha\lambda}{2\pi l\cos\theta}\cos\frac{2\pi}{\lambda}\,(Vt - l + z\cos\theta);\end{aligned}\right\} \tag{85}$$

and between the limits corresponding to $+b$ and $-b$,

$$\delta = \frac{\alpha\lambda}{\pi l \cos\theta} \sin\frac{2\pi b\cos\theta}{\lambda} \sin 2\pi \frac{Vt - l}{\lambda}. \tag{86}$$

The maximum displacement is therefore

$$\delta = \frac{\alpha\lambda}{\pi l \cos\theta} \sin\frac{2\pi b\cos\theta}{\lambda}. \tag{87}$$

70. 1°. The above value of the displacement will vary with b, θ, λ, and l. When, as in sound, λ is very great as compared with b, $\frac{2\pi b\cos\theta}{\lambda}$ will be so small that the arc may be substituted for the sine without material error, and

$$\delta = \frac{2\alpha b}{l}, \tag{88}$$

which is independent of θ.

2°. When λ is much smaller than b, as in the case of light, we have, when $\cos\theta$ is very small, and θ therefore differs but little from 90°, again

$$\delta = \frac{2\alpha b}{l}. \tag{89}$$

At other points, where θ is not great and $\cos\theta$ not small, the resultant displacement becomes equal to zero when

$$\frac{2\pi b\cos\theta}{\lambda} = \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \quad \text{etc.};$$

that is, when $\quad \cos\theta = \pm\frac{\lambda}{2b}, \quad \pm\frac{2\lambda}{2b}, \quad \pm\frac{3\lambda}{2b}, \quad$ etc.

The greatest resultant displacement, other than that indicated above, will be found by making in Eq. (87),

$$\sin\frac{2\pi b\cos\theta}{\lambda} = \pm 1, \tag{90}$$

and it will be equal to $\qquad \dfrac{\alpha\lambda}{\pi l \cos\theta};$

and, since the intensity of the sensation is directly proportional to

the square of the maximum displacements, we will have the relation of the intensities,

$$\frac{a^2\lambda^2}{\pi^2 l^2 \cos^2\theta} : \frac{4a^2b^2}{l^2} = \lambda^2 : 4\pi^2 b^2 \cos^2\theta = \frac{\lambda^2}{4\pi^2 b^2 \cos^2\theta} : 1. \quad (91)$$

71. In acoustics it will be shown that the wave lengths corresponding to audible sounds will vary from $\frac{1140'}{20} = 57'$ to $\frac{1140'}{40000}$ $= \frac{1}{3}$ of an inch, and therefore there will be no point exterior to an aperture where the displacement will not occur, and hence the corresponding sound be heard. In light, the wave lengths vary between .000026 and .000017 of an inch, and there will be, according to the 2° case, alternations of light and darkness surrounding the central line drawn from the place of original disturbance to the centre of the aperture. These zones are called Huyghens' zones, and will be again referred to in the subject of diffraction.

72. ***Diffusion and Decay of Kinetic Energy.*** The displacement of any molecule due to wave motion of a given wave length is independent of the periodic time, and, since the orbits of the molecules are described in equal times when they arise from a given periodic motion, they will be directly proportional to the displacements or any other homologous lines. The velocities, then, of the moving molecules being represented by v, their kinetic energies will be represented by $\frac{mv^2}{2}$. Then, because these energies are transmitted without appreciable loss from the molecules of one surface to those of another, we will have the energies of the molecules of the two homologous surfaces,

$$4\pi r^2 \cdot \frac{mv^2}{2} = 4\pi r'^2 \cdot \frac{mv'^2}{2}, \quad \text{or} \quad \frac{mv^2}{2} \cdot r^2 = \frac{mv'^2}{2} \cdot r'^2; \quad (92)$$

that is, $\frac{mv^2}{2} : \frac{mv'^2}{2} :: r'^2 : r^2$, or varying according to the law of the inverse square of the distance. Similarly, we will have

$$\delta'' r'' = \delta' r', \quad (93)$$

or the maximum displacements inversely proportional to the distances to which the disturbance has been propagated.

73. ***Reflection and Refraction.*** It is difficult to conceive, satisfactorily, in what manner the molecules belonging to two media of different elasticity and density are arranged with respect to each other in or near the bounding surface which separates them. When they occupy positions of relative rest, the elastic forces must be mutually counterbalanced and must be equal to those affecting the molecules within the media. We may assume the two media to have different densities and elasticities, and the relative positions of the molecules near the separating surface to be determined by the action of the equilibrating molecular forces. But when a disturbance arising in one of the media reaches the surface, the molecules of the second medium must, in general, have motions and displacements different from those of the first. If we consider alone the difference in density of the molecules of the media, we perceive that the energy in the incident wave will not be wholly given up by the molecules to their neighbors in the new medium. In either case, whether the molecules have greater or less density, a return wave will originate in the incident medium, analogous to the reflected motion in the impact of elastic balls. Again, if the elasticity of the media be different, the elastic forces for equal displacements will be different, and thus cause a return wave in the incident medium. We may therefore assume, for the present, that, owing to the different elasticities or densities, or both, there will be, in general, a separation of the incident wave whenever it meets a surface separating two media of different density and elasticity. The fact of such a separation is experimentally demonstrated in the phenomena of sound and light. The velocity of wave propagation will be shown to be a function of the elasticity and density of the medium, and therefore the waves, in general, will proceed in the two media with different velocities.

74. The *plane of incidence* is that plane which is normal to the deviating surface and to the wave front.

The *plane of reflection* is normal to the deviating surface and to the reflected wave front; it coincides with the plane of incidence.

The *plane of refraction* is normal to the refracted wave front and to the deviating surface.

75. ***Diverging, Converging, and Plane Waves.*** When the energy of molecular disturbance is distributed among

the molecules, upon an increasing wave front, the wave is said to be *diverging;* when among those of a decreasing wave front, a *converging* wave; and when among those of an unchanged wave front, a *plane* wave. An indefinitely small portion of the front of any diverging wave, taken at a correspondingly great distance from the origin, may, without sensible error, be considered as coinciding throughout with the tangent plane to the wave front, and considered as a plane wave. The molecules of a plane wave at any assumed position are animated by equal parallel displacements, and undergo all their phases while the plane wave advances a distance equal to the wave length, measured in a direction perpendicular to the plane.

76. Differentiating Eq. (67), we have

$$\frac{d^2\delta}{dt^2} = -\frac{4\pi^2 V^2}{\lambda^2}\,\delta. \tag{94}$$

Multiplying both members by m, the mass of the molecule, and replacing $m\dfrac{d^2\delta}{dt^2}$ by its equal $U\delta$, the intensity of the elastic force developed by the displacement δ, we have

$$U\delta = m\frac{d^2\delta}{dt^2} = -\frac{4\pi^2 m}{\lambda^2}V^2\delta; \tag{95}$$

whence,

$$U = -\frac{4\pi^2 m}{\lambda^2}V^2. \tag{96}$$

Hence, *when a plane wave is propagated without alteration in a homogeneous medium, its velocity of propagation is directly proportional to the square root of the elastic force developed by the displacement of its molecules.*

77. ***Reflection and Refraction of Plane Waves.*** Let the incident plane wave AC (Fig. 11) meet the deviating surface at all points, in succession, from A to B. Let V and λ be the velocity of wave propagation and the wave length in the medium of incidence, and V' and λ' those in the medium of intromittance. Let AB $= ds$, and CB $= V dt$. While the disturbance in the incident wave is moving from C to B, the disturbance from A as a centre will proceed in all directions in the medium of incidence,

and be found, at the instant considered, upon the hemisphere whose radius is AD $=$ CB $= V\,dt$, and in the medium of intromittance on the hemisphere whose radius is AD$' = V'\,dt$.

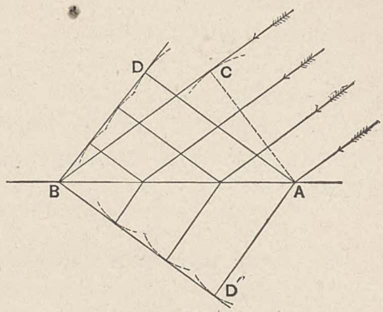

Figure 11.

Each point in the line AB will, in like manner, become in succession a new centre of disturbance, sending secondary waves into the media of incidence and of intromittance, whose radii will, at the instant the incident wave reaches B, be equal to V and V' multiplied by the interval of time elapsing between the instant of arrival of the wave front at the centre considered and that of its arrival at B. The surface through B, which is tangent to all the reflected pulses, may be taken as the front of the reflected wave, for it will contain more energy than any other surface of equal area in the incident medium. Similarly, the surface through B tangent to all the refracted pulses will contain more energy than any other of equal area in the medium of intromittance, and may be taken as the front of the refracted wave at this instant. These surfaces are readily seen to be planes; hence, denoting the angle CAB = ABD by ϕ, and ABD$'$ by ϕ', we will have

$$ds \sin \phi = V\,dt, \qquad ds \sin \phi' = V'\,dt\,; \tag{97}$$

from which we obtain

$$\sin \phi = \frac{V}{V'} \sin \phi' = \mu \sin \phi', \tag{98}$$

which is known as Snell's law of the sines; μ is called the index of refraction.

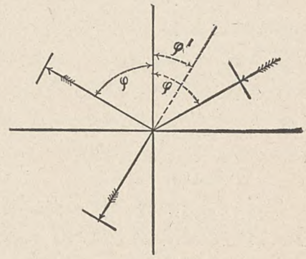

Figure 12.

78. The angles ϕ and ϕ' made by the wave fronts with the deviating surface are, respectively, equal to the angles made by the normals to the incident and refracted waves with the normal to the deviating surface, and

are called *angles of incidence* and *refraction.* The angles of incidence and refraction are measured from the normal to the deviating surface on the side of the medium of incidence to the normal of the incident wave, and to that of the refracted wave produced back into the medium of incidence.

The angle of reflection is measured from the normal to the deviating surface to the normal to the reflected wave front, and is therefore negative. In the reflected wave, since the velocity of wave propagation is unchanged, μ is equal to unity, and Eq. (98) becomes

$$\sin \phi = - \sin \phi'. \tag{99}$$

79. These principles may, in ordinary cases, then be summarized as follows:

1°. The planes of incidence, reflection, and refraction are coincident.

2°. The sine of the angle of incidence is equal to the index of refraction or of reflection multiplied by the sine of the angle of refraction or of reflection.

The modifications which take place in polarized light will be referred to hereafter in physical optics.

80. We see from Art. 77 that the reflected and refracted waves are plane when an incident plane wave meets a plane deviating surface. It is evident also, from the construction, that the reflected rays are all normal to a plane NN′ symmetrical with MM′ with reference to OX; and that the incident and reflected rays are directed from their corresponding planes towards the deviating surface. The refracted rays are normal to a plane RR′ on the same side of the deviating surface as the incident wave, and are also directed towards that surface.

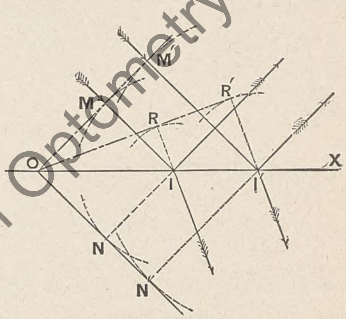

Figure 13.

81. ***General Construction of the Reflected and Refracted Waves.*** Let the deviating surface AB (Fig. 14) be any whatever, and the rays proceed from any origin O; take, in

the medium of incidence, any spherical surface SS′, with centre at O, as the incident diverging wave; then, from all points I, I′, I″, etc., of AB, describe spheres, whose radii are equal to the intercepts of the rays between SS′ and AB. If, now, tangent planes be drawn to the deviating surface at I, I′, I″, etc., and to the surface SS′ at the corresponding points s, s', s'', etc., each pair of tangent planes will determine, by their intersection, a right line, through which if a plane be passed tangent to the corresponding sphere on the other side of the deviating surface, it will be symmetrical with the infinitesimal surface of SS′ at s with respect to that of AB at the point I; and similarly for the other points. By continuity, these points of tangency may be considered as forming the envelope of the reflected wave. The direction of the reflected rays is found by joining these points with I, I′, I″, etc., and extending the lines toward and beyond the deviating surface.

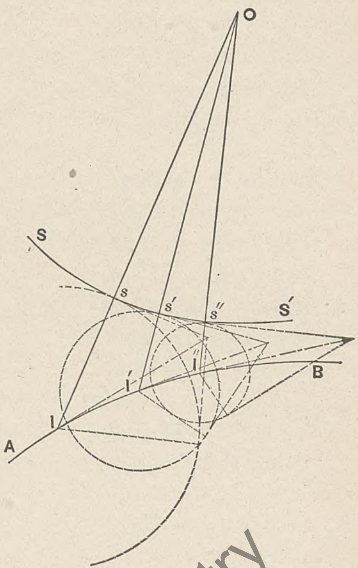

Figure 14.

82. By the proper modification of the radii due to the value of μ, the index of refraction, the envelope of the refracted wave and the direction of the refracted rays may be constructed.

83. Considering the reflected wave as a new incident wave, the new reflected wave, by another deviating surface, can be constructed by an application of the above principles; and since reflection may be considered as refraction whose index is -1, the principle may be generally stated, that any number of reflections and refractions may be replaced by a single refraction at a supposable deviating surface with a properly modified index of refraction.

84. Let DEF (Fig. 15) be any incident wave whose rays are not necessarily parallel; MNP any deviating surface. At some subsequent time t the incident wave will occupy some position such as ABG, FG being equal to $EB = DA = Vt$. By the principle

established above, abg will be the enveloping surface of the reflected wave corresponding to ABG, and $a_1b_1g_1$ that of the refracted wave, and both will be concurrent, that is, the phases of the molecular motions on them will be similar; gPG', bNB', aMA' will be the reflected, and g_1PG_1, b_1NB_1, a_1MA_1 the refracted rays.

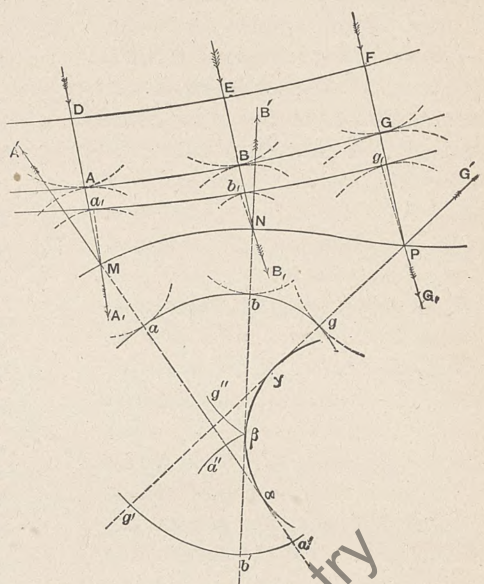

Figure 15.

85. Prolong the consecutive rays of either the reflected or refracted waves, say the reflected wave abg, until they meet two and two; they will be tangent to the surface $\alpha\beta\gamma$, which is the evolute of abg. Since the reflected rays are all normal to abg, this evolute will correspond to any other position of the reflected wave, also. The surface of which $\alpha\beta\gamma$ is a generatrix is in optics called the *caustic surface*. It is evident that the points of this caustic are not concurrent, because their distances, being equal to the radii of curvature of abg from the reflected wave, are themselves unequal; and points, in order to be concurrent, must be at equal distances from the wave surface. Whether the caustic be real or virtual, the displacements of its molecules being either due to that of two rays, or apparently so, the energy of the molecules, and hence the resulting sensation, will be greater than that due to but one ray.

86. When the evolute $\alpha\beta\gamma$ is known, the various possible positions of the reflected wave can readily be determined. In the ordinary cases considered in optics, the surfaces abg are those of revolution; the caustic is then also a surface of revolution. Suppose abg to be one of the generatrices of the reflected wave, considered as a surface of revolution, and $\alpha\beta\gamma$ to be its evolute; then, by

the property of the evolute, if the tangent $a\alpha a'$ be caused to roll on $\alpha\beta\gamma$, each point of this tangent will describe one of the sections of the reflected wave. Thus, $a'b'g'$, $a''\beta g''$, and abg are such sections; the second of these being of two nappes, tangent to each other and normal to the evolute at the point β.

87. The principle that the rays, after the wave has been subjected to any number of reflections and refractions, are all normal to a theoretically determinable surface, and consequently to a series of surfaces, of which any two intercept the same length on all the rays, is principally applicable to the determination of caustic surfaces, and to the formation of optical images, and will therefore be further discussed in that branch of the subject.

88. ***Utility of Considering the Propagation of the Disturbance by Plane Waves.*** In a homogeneous medium, the arbitrary displacement of a molecule gives rise to elastic forces whose intensities depend on the degree and the direction of the displacements, and whose directions are not, in general, those of the displacements. In Art. (35) we have seen that the displacements must be made only in exceptional directions, in order that the elastic forces varying directly with the degree of the displacement should be wholly in those directions. Should the orbit of the displaced molecule be curvilinear, it is evident that, at each point of its path, the elastic forces developed would vary both in direction and intensity, and thus the general problem becomes one of extreme intricacy.

89. If, however, it be possible to limit the discussion to that of molecules in the same plane, all actuated by equal and parallel displacements, the variation as to direction of the elastic forces may, perhaps, be eliminated. It has been shown, Art. 76, that when a plane wave is propagated without alteration in a homogeneous medium, the velocity of propagation is directly proportional to the square root of the elastic force developed by the displacement. Hence the importance of deducing from the general equations (18) the corresponding equations applicable to the vibratory motions propagated by plane waves.

90. At the time t let r be the distance of the plane wave, in a homogeneous medium, from the origin of co-ordinates; ε the displacement of the molecules whose co-ordinates are x, y, z; ξ, η, ζ,

the projections of ε on the rectangular co-ordinate axes ; and α, β, γ, the angles made by the displacement with the axes, respectively.

We then have $$\varepsilon = \delta \sin \frac{2\pi}{\lambda}(Vt - r); \tag{100}$$

$$\left.\begin{aligned} \xi &= \delta \cos \alpha \sin \frac{2\pi}{\lambda}(Vt - r); \\ \eta &= \delta \cos \beta \sin \frac{2\pi}{\lambda}(Vt - r); \\ \zeta &= \delta \cos \gamma \sin \frac{2\pi}{\lambda}(Vt - r). \end{aligned}\right\} \tag{101}$$

Let $r + \Delta r$ be the distance of the plane at a subsequent instant from the origin, and l, m, n, the angles made by the normal to the plane with the axes, then

$$r = x \cos l + y \cos m + z \cos n, \tag{102}$$

$$\Delta r = \Delta x \cos l + \Delta y \cos m + \Delta z \cos n. \tag{103}$$

From Eq. (101) we have

$$\left.\begin{aligned} \xi + \Delta\xi &= \delta \cos \alpha \sin \frac{2\pi}{\lambda}(Vt - r - \Delta r) \\ &= \delta \cos \alpha \left[\sin \frac{2\pi}{\lambda}(Vt - r) \cos \frac{2\pi}{\lambda}\Delta r \right. \\ &\qquad \left. - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \right]; \end{aligned}\right\} \tag{104}$$

from which, and similarly for the axes y and z, we have

$$\left.\begin{aligned} \Delta\xi &= \delta \cos \alpha \left[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \right. \\ &\qquad \left. - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \right], \\ \Delta\eta &= \delta \cos \beta \left[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \right. \\ &\qquad \left. - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \right], \\ \Delta\zeta &= \delta \cos \gamma \left[\sin \frac{2\pi}{\lambda}(Vt - r)\left(\cos \frac{2\pi}{\lambda}\Delta r - 1\right) \right. \\ &\qquad \left. - \cos \frac{2\pi}{\lambda}(Vt - r) \sin \frac{2\pi}{\lambda}\Delta r \right]. \end{aligned}\right\} \tag{105}$$

Substituting these values in Eqs. (18), and, since the medium is homogeneous, the sums arising from the substitution of the second part of the values of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, and which are of the form

$$\left.\begin{array}{l}
\Sigma\mu\,\phi(r)\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta y^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta x^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta z^2}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,\\[2mm]
\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin\frac{2\pi}{\lambda}\Delta r,
\end{array}\right\} \tag{106}$$

all reduce to zero, because they are formed of terms which, two and two, are equal, with contrary signs; for, to the values of Δx, Δy, Δz, equal, with contrary signs, correspond values of Δr which are also equal and have contrary signs. Then, replacing $\cos\frac{2\pi}{\lambda}\Delta r$ by its equal, $1 - 2\sin^2\frac{\pi}{\lambda}\Delta r$, and $\delta\sin\frac{2\pi}{\lambda}(Vt - r)$ by its equal ε, Eqs. (18) become, for plane waves,

$$X_1 = -\frac{X}{2m} = \cos\alpha\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r$$
$$+ \cos\beta\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r + \cos\gamma\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r,$$

$$Y_1 = -\frac{Y}{2m} = \cos\alpha\,\Sigma\mu\,\psi(r)\frac{\Delta x\,\Delta y}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r.$$
$$+ \cos\beta\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta r$$
$$+ \cos\gamma\,\Sigma\mu\,\psi(r)\frac{\Delta y\,\Delta z}{r^2}\sin^2\frac{\pi}{\lambda}\Delta r,$$

$$Z_1 = -\frac{Z}{2m} = \cos\alpha\, \Sigma\mu\, \psi(r) \frac{\Delta x\, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta r$$

$$+ \cos\beta\, \Sigma\mu\, \psi(r) \frac{\Delta y\, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta r$$

$$+ \cos\gamma\, \Sigma\mu \left[\phi(r) + \psi(r) \frac{\Delta z^2}{r^2} \right] \sin^2 \frac{\pi}{\lambda} \Delta r. \tag{107}$$

91. The conditions for the propagation of the plane wave without change are

$$\frac{X_1}{\cos\alpha} = \frac{Y_1}{\cos\beta} = \frac{Z_1}{\cos\gamma} = U_1. \tag{108}$$

Substituting, in Eq. (107), for Δr its equal,

$$\Delta x \cos l + \Delta y \cos m + \Delta z \cos n = \Delta r, \tag{109}$$

and substituting in Eqs. (108) the values of X_1, Y_1, Z_1, thus obtained, we will have two relations which, with

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \tag{110}$$

will enable us to determine the angles α, β, γ, which the displacement should make with the axes, in order that the propagation of the plane wave may be possible.

92. Because of the equality of the coefficients of $\cos\beta$ and $\cos\alpha$ in the first and second of Eqs. (107), and of $\cos\beta$ and $\cos\gamma$ in the third and second, and of $\cos\alpha$ and $\cos\gamma$ in the third and first, we can, by substitutions and reductions similar to those employed in Art. 33, deduce corresponding principles, and hence determine that, for each direction of the plane wave, there correspond, for the molecular displacements, three rectangular directions such that the plane wave may be propagated without change, and that these three directions are parallel to the three axes of an ellipsoid whose equation is,

$$
\left.\begin{aligned}
&x^2 \, \Sigma\mu \left[\phi(r) + \psi(r) \frac{\Delta x^2}{r^2} \right] \sin^2 \frac{\pi}{\lambda} \Delta r \\
&+ y^2 \, \Sigma\mu \left[\phi(r) + \psi(r) \frac{\Delta y^2}{r^2} \right] \sin^2 \frac{\pi}{\lambda} \Delta r \\
&+ z^2 \, \Sigma\mu \left[\phi(r) + \psi(r) \frac{\Delta z^2}{r^2} \right] \sin^2 \frac{\pi}{\lambda} \Delta r \\
&+ 2xy \, \Sigma\mu \, \psi(r) \frac{\Delta x \, \Delta y}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta r \\
&+ 2xz \, \Sigma\mu \, \psi(r) \frac{\Delta x \, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta r \\
&+ 2yz \, \Sigma\mu \, \psi(r) \frac{\Delta y \, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta r
\end{aligned}\right\} = 1. \qquad (111)
$$

This is called either the *inverse ellipsoid* or the *ellipsoid of polarization*. Having also the relation expressed in Eq. (109), we see that the coefficients of Eq. (111) depend upon the angles l, m, n, which determine the direction of the plane wave, upon certain constants which define the constitution of the medium, and upon the wave length. The velocity of propagation is inversely proportional to the length of that axis of the ellipsoid to which the molecular displacements are parallel.

93. ***Relation between the Velocity of Wave Propagation of Plane Waves and the Wave Length in Isotropic Media.*** All directions being identical in isotropic media, we will assume the plane wave normal to the axis of x. We then have $\Delta r = \Delta x$, and

$$
\left.\begin{aligned}
&\Sigma\mu \, \psi(r) \frac{\Delta x \, \Delta y}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta x = 0, \\
&\Sigma\mu \, \psi(r) \frac{\Delta x \, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta x = 0, \\
&\Sigma\mu \, \psi(r) \frac{\Delta y \, \Delta z}{r^2} \sin^2 \frac{\pi}{\lambda} \Delta x = 0;
\end{aligned}\right\} \qquad (112)
$$

and Eqs. (107) reduce to

$$\left.\begin{aligned}
X_1 &= \cos\alpha\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x, \\
Y_1 &= \cos\beta\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x, \\
Z_1 &= \cos\gamma\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x;
\end{aligned}\right\} \quad (113)$$

and the equation of the ellipsoid to

$$\left.\begin{aligned}
&x^2\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x \\
+\, &y^2\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x \\
+\, &z^2\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta z^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x
\end{aligned}\right\} = 1; \quad (114)$$

and, since all directions perpendicular to the axis of x are identical with reference to the plane of the wave, we have $\Delta y = \Delta z$, and Eq. (114) of the ellipsoid becomes one of revolution about the axis of x. Whence, we conclude that, in an isotropic medium, a plane wave normal to a given direction can be propagated without change, whenever the molecular displacement is parallel or perpendicular to this direction. To any one direction of normal propagation in such a medium, there corresponds an infinite number of waves with transversal vibrations, having the same velocity, and but *one* wave with longitudinal vibrations whose velocity is different from those with transversal vibrations.

94. For the wave with longitudinal vibrations, we have

$$\alpha = 0, \qquad \beta = \gamma = 90^\circ,$$

$$\left.\begin{aligned}
U_1 = X_1 &= \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x, \\
Y_1 &= Z_1 = 0;
\end{aligned}\right\} \quad (115)$$

and, from Eqs. (96) and (107), we have

$$V^2 = \frac{\lambda^2}{2\pi^2}\, \Sigma\mu \left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right] \sin^2\frac{\pi}{\lambda}\Delta x. \quad (116)$$

In Acoustics, it will be shown that sound is due to longitudinal vibrations of the medium. This equation will then be applicable in all cases of sound arising from such vibrations, and will be referred to, in that branch of the subject. In Optics, it will be shown that transversal vibrations only are efficacious in producing light.

95. For waves with transversal vibrations in isotropic media, the velocity is independent of the direction of the displacement. We can then suppose the displacement parallel to the axis of y, and thus have

$$\alpha = \gamma = 90^\circ, \qquad \beta = 0,$$

$$\left.\begin{aligned} X_1 &= Z_1 = 0, \\ Y_1 &= \Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x = U_1\,; \end{aligned}\right\} \tag{117}$$

and $$V^2 = \frac{\lambda^2}{2\pi^2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x. \tag{118}$$

This equation is applicable in light, for the determination of wave velocity in isotropic and homogeneous media, and will be used hereafter in determining the velocity of light propagation.

By developing $\sin^2\frac{\pi}{\lambda}\Delta x$ into a series, we find

$$\sin^2\frac{\pi}{\lambda}\Delta x = \frac{\pi^2}{\lambda^2}\Delta x^2 - \frac{1}{3}\cdot\frac{\pi^4}{\lambda^4}\Delta x^4 + \frac{2}{45}\cdot\frac{\pi^6}{\lambda^6}\Delta x^6 - \frac{1}{315}\cdot\frac{\pi^8}{\lambda^8}\Delta x^8 + \text{etc.}$$

Substituting this in Eq. (118), we obtain

$$V^2 = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \frac{d}{\lambda^6} + \text{etc.}, \tag{119}$$

in which $a, b, c, \ldots.$ have for values,

$$\left.\begin{aligned} a &= \quad \frac{1}{2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^2, \\ b &= -\frac{\pi^2}{6}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^4, \\ c &= \quad \frac{\pi^4}{45}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta y^2}{r^2}\right]\Delta x^6, \\ d &= - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{120}$$

These constants depend only on the *constitution* of the medium, and decrease very rapidly in value, for Δx is always a very small quantity. If the wave length be not excessively small, if it surpasses a certain value which observation only can determine, the terms of the second member of Eq. (119) will have very rapidly decreasing values, and we will obtain an expression approximately near to V^2 by taking only the first few terms. Hence, a must be positive, and, since observation shows that the most refrangible rays are those of the shortest wave length, and that, as a consequence, V decreases with λ, b is necessarily negative.

96. Hence, in isotropic media, the elasticity being uniform in all directions, the form of the wave surface will be spherical, and when the displacements are longitudinal, its radius at the unit time from the epoch will be the value of V obtained from Eq. (116); when the displacements are transversal, the radius will be the value of V in Eq. (118). The former relates wholly to waves of sound, and the latter to those of light.

The subsequent discussion will now apply to transversal vibrations alone, and the conclusions derived belong therefore to the transmission of light undulations.

Experiment shows that the media which transmit the waves of light are not in general isotropic, and as a consequence the form of the wave surface will not be spherical. We will, therefore, now seek the form of this surface in the general case, and make use of the properties of plane waves for this purpose.

97. ***Plane Waves in a Homogeneous Medium of Three Unequal Elasticities in Rectangular Directions.*** In the plane wave, the following conclusions have been deduced:

1°. The displacements of the molecules, in each position of the same plane wave, must be rectilineal and parallel to each other and to their original directions.

2°. The elastic forces developed by these displacements must be either in the directions of the displacements or alone efficacious in these directions.

3°. The propagation of the plane wave unaltered is then possible.

These conclusions involve, as consequences, a constancy of ve-

locity of propagation when the plane wave is unchanged in direction, and a variation in the velocity as the direction is changed. Hence, if the elasticities of a homogeneous medium differ in all directions, and we suppose plane waves, having all possible positions, originate at any point m of an indefinite medium, these plane waves, at the end of a unit of time, will be at different distances from m. The surface which is the envelope of all these plane waves at this instant is called the *wave surface.*

98. Let
$$a > b > c$$
be the principal axes of elasticity of such a medium. Then a^2, b^2, c^2, will measure the *elastic forces* developed in these directions by a displacement equal to unity, and any of the surfaces of elasticity heretofore determined can be used to obtain the elastic forces developed by an equal displacement in the direction of the corresponding radius vector of the surface. The velocity of wave propagation being proportional to the square root of the elastic force, Eq. (96), its value can be found when the elastic force due to the displacement in any direction is known.

99. Fresnel made use of the single-napped surface of elasticity whose equation is
$$a^2x^2 + b^2y^2 + c^2z^2 = r^4; \tag{121}$$
but for plane waves, the inverse ellipsoid of elasticity or first ellipsoid,
$$a^2x^2 + b^2y^2 + c^2z^2 = 1, \tag{E}$$
together with its reciprocal ellipsoid,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{W}$$
can be more readily used, because of its better known properties. The squares of the semi-axes of (W) and of the reciprocals of (E) are the principal elasticities of the medium.

100. There are two cases to consider:

1°. *The plane of the wave contains two of the principal axes,* and hence is one of the *principal planes* of the medium. The plane cuts the ellipsoids in ellipses whose semi-axes are either two of the principal axes. Whatever be the direction and amount of

the displacement, it may be replaced by its components in the direction of the axes proportional to cos α and sin α, α being the angle made by the displacement with either axis.

Considering these separately, we see: 1°, that each will communicate to the molecules in the adjacent plane analogous rectilineal motions which will be propagated without alteration of direction; 2°, that the elasticities, and hence the velocities of propagation which belong to these two, are different, and that after a time there will be two series of molecules situated in parallel planes, parallel also to the primitive plane, which will contain all of the original energy; 3°, that the vibrations of the molecules in these two plane waves will be at right angles to each other.

101. 2°. *The plane wave is any whatever.* The sections of the ellipsoids will be ellipses, but will not in general contain either of the axes of the ellipsoids. There will then be no direction of the displacement that can give a resultant elastic force in the direction of the displacement. It is, therefore, essential for a rectilineal oscillation of the molecule and for a consecutive transmission of this oscillation, that there should be no tendency of the rectilineal displacement to be deflected on either side, but that the line of the resultant force should be projected upon the displacement. As it is not in general in the plane, but oblique to it, it can be resolved into two components: one normal to the plane, which is not effective in light undulations; and the other, which is alone efficacious, in the direction of the displacement. In each elliptical section there are two such directions, which are named *singular directions*, and which are perpendicular to each other. Assume any plane section through the centre of (E); the elasticity measured by the squares of the reciprocals of the radii-vectores is the same to the right and left for the two axes of the section, and is the same only for them. Through either of the axes pass the normal plane to the section; it will cut all the parallel plane sections in their homologous axes. With reference to this normal plane, the radii-vectores, and therefore the elasticities of each section, are symmetrical. Hence, if the displacement be along one of the axes of the section, the total elastic force will be in the normal plane, and will be projected on the axis of the section. And since the ellipsoid semi-diameters are inversely as the velocities of propagation, the recipro-

cals of the axes will measure the velocities of wave propagation. Hence is established the fact that for each section there are two of these singular directions, and that they are rectangular. These two singular directions perform the same function for the vibrations of the plane wave as do the axes of elasticity themselves when the plane wave contains them. Each vibration is replaced by two others in the direction of the singular directions, and these two components proceed in the medium without change of direction, but with different velocities, so that there are then, in the general case, two plane waves parallel to each other and to the original plane wave. If α be the angle made by the displacement with one of the axes, the component displacements will be proportional to $\cos \alpha$ and $\sin \alpha$, and the elastic intensities to $\cos^2 \alpha$ and $\sin^2 \alpha$. Whatever may be the direction of the original supposed vibration in the plane wave, the two plane waves which replace it are always the two above designated.

102. If the plane of the wave coincides with either of the circular sections of the ellipsoid, the plane wave will be propagated without alteration, whatever be the direction of the displacement, with a velocity equal to b, the reciprocal of the mean semi-axis of the ellipsoid.

103. ***The Double-Napped Surface of Elasticity.*** If through the centre of (E) we pass any plane, and on the normal to the section at the centre set off distances inversely proportional to the semi-axes of the section, the locus of all these pairs of points is called the *double-napped surface of elasticity*. For, each radius vector measures the velocity of propagation of one of the plane waves, arising from a displacement in the plane of section, and the square of each of these normal velocities is the measure of the elastic force developed by the component displacement along the axes of the section.

104. If through each of the points so determined planes be passed parallel to the corresponding plane of section, the envelope of all these planes will be, by definition, the wave surface. Hence, the latter can be constructed by points from this surface of elasticity.

105. To get the polar equation of the latter surface, let us take for co-ordinate axes the principal axes of the medium ; let l, m, n,

be the angles made by the normal to the plane wave with these axes, x, y, z, respectively; α, β, γ, those which one of the axes of the ellipse of section make with the same axes; then we have

$$\cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n = 0. \tag{122}$$

The elastic force developed by a displacement parallel to the axis of section is projected on the plane of the wave parallel to this displacement, and its components are

$$X = a^2\cos\alpha, \quad Y = b^2\cos\beta, \quad Z = c^2\cos\gamma. \tag{123}$$

The cosines of the angles which this elastic force makes with the axes are then proportional to these values. An auxiliary right line perpendicular to the direction of the elastic force and to the displacement will lie in the plane of the wave, and if u, v, w, be the angles which it makes with the axes, we will have

$$\left.\begin{aligned} a^2\cos\alpha\cos u + b^2\cos\beta\cos v + c^2\cos\gamma\cos w &= 0,\\ \cos\alpha\cos u + \cos\beta\cos v + \cos\gamma\cos w &= 0,\\ \cos l\cos u + \cos m\cos v + \cos n\cos w &= 0. \end{aligned}\right\} \tag{124}$$

Representing the velocity of propagation of the plane wave by V, we have

$$V^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma. \tag{125}$$

Combining the above equations, and eliminating the quantities α, β, γ, u, v, w, we will have an equation containing V, l, m, n, which will be that of the surface required. To eliminate u, v, w, we will make use of the method of indeterminate coefficients; thus, multiply Eqs. (124) by B, A, and unity, respectively, add the three resulting equations, and from the conditions for B and A that the coefficients of $\cos v$ and $\cos w$ shall reduce to zero, we will have

$$\left.\begin{aligned} (A + Ba^2)\cos\alpha + \cos l &= 0,\\ (A + Bb^2)\cos\beta + \cos m &= 0,\\ (A + Bc^2)\cos\gamma + \cos n &= 0. \end{aligned}\right\} \tag{126}$$

Multiply these by $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, respectively, add, and reduce by Eqs. (122), (125); we will have

$$A + BV^2 = 0. \tag{127}$$

Substitute this value of A in Eqs. (126), and we have

$$\left.\begin{aligned}\cos l &= B\,(V^2 - a^2)\cos\alpha,\\ \cos m &= B\,(V^2 - b^2)\cos\beta,\\ \cos n &= B\,(V^2 - c^2)\cos\gamma.\end{aligned}\right\} \qquad (128)$$

From which we get

$$\left.\begin{aligned}\frac{\dfrac{\cos l}{V^2-a^2}}{\cos\alpha} = \frac{\dfrac{\cos m}{V^2-b^2}}{\cos\beta} = \frac{\dfrac{\cos n}{V^2-c^2}}{\cos\gamma}\\ = \sqrt{\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}}.\end{aligned}\right\} \qquad (129)$$

Replacing $\cos\alpha$, $\cos\beta$, $\cos\gamma$, in Eq. (122), by their proportional quantities, $\dfrac{\cos l}{V^2-a^2}$, $\dfrac{\cos m}{V^2-b^2}$, $\dfrac{\cos n}{V^2-c^2}$, we have

$$\frac{\cos^2 l}{V^2-a^2} + \frac{\cos^2 m}{V^2-b^2} + \frac{\cos^2 n}{V^2-c^2} = 0, \qquad (130)$$

the polar equation of the double-napped surface of elasticity, in which V is any radius vector.

106. ***The Wave Surface.*** Through any point of the surface of elasticity pass a plane perpendicular to the radius vector at that point, and let r, λ, μ, ν, be the polar co-ordinates of any point of the plane. The equation of the plane will be

$$\cos l\cos\lambda + \cos m\cos\mu + \cos n\cos\nu = \frac{V}{r}. \qquad (131)$$

We have also, as equations of condition,

$$\cos^2 l + \cos^2 m + \cos^2 n = 1, \qquad (132)$$

$$\frac{\cos^2 l}{V^2-a^2} + \frac{\cos^2 m}{V^2-b^2} + \frac{\cos^2 n}{V^2-c^2} = 0. \qquad (133)$$

The wave surface is the enveloping surface of the planes given by Eq. (131), and its equation can be determined by eliminating V, l, m, n, and finding an equation between r, λ, μ, ν. To do this, differentiate Eqs. (131), (132), (133), regarding $\cos l$ and $\cos m$ as independent variables, and we will have

$$\left.\begin{aligned}
&\cos\lambda + \cos\nu\,\frac{d\cos n}{d\cos l} = \frac{1}{r}\cdot\frac{dV}{d\cos l},\\
&\cos l + \cos n\,\frac{d\cos n}{d\cos l} = 0,\\
&\frac{\cos l}{V^2-a^2} + \frac{\cos n}{V^2-c^2}\cdot\frac{d\cos n}{d\cos l}\\
&\quad = \frac{dV}{d\cos l}\,V\left[\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}\right];
\end{aligned}\right\} \quad (134)$$

$$\left.\begin{aligned}
&\cos\mu + \cos\nu\,\frac{d\cos n}{d\cos m} = \frac{1}{r}\cdot\frac{dV}{d\cos m},\\
&\cos m + \cos n\,\frac{d\cos n}{d\cos m} = 0,\\
&\frac{\cos m}{V^2-b^2} + \frac{\cos n}{V^2-c^2}\cdot\frac{d\cos n}{d\cos m}\\
&\quad = \frac{dV}{d\cos m}\,V\left[\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}\right].
\end{aligned}\right\} \quad (135)$$

107. To eliminate $\frac{d\cos n}{d\cos l}$, $\frac{d\cos n}{d\cos m}$, $\frac{dV}{d\cos l}$, $\frac{dV}{d\cos m}$, multiply Eqs. (134) by 1, A, and $-B$, respectively, add the resulting equations together, and perform the same operations on Eqs. (135). Supposing the indeterminate quantities A and B to have such values as will make the coefficients of $\frac{d\cos n}{d\cos l}$, $\frac{d\cos n}{d\cos m}$, $\frac{dV}{d\cos l}$, $\frac{dV}{d\cos m}$, equal to zero, we will have

$$\left.\begin{aligned}
&\cos\lambda + A\cos l = B\,\frac{\cos l}{V^2-a^2},\\
&\cos\mu + A\cos m = B\,\frac{\cos m}{V^2-b^2},\\
&\cos\nu + A\cos n = B\,\frac{\cos n}{V^2-c^2},\\
&\frac{1}{r} = BV\left[\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}\right].
\end{aligned}\right\} \quad (136)$$

Multiply the first three of the above equations by $\cos l$, $\cos m$,

and cos n, respectively, add the resulting equations, and reduce by Eqs. (131), (132), (133); we will have

$$A + \frac{V}{r} = 0. \tag{137}$$

Squaring the first three of Eqs. (136), and adding, we get, after reduction

$$1 + 2A\frac{V}{r} + A^2 = \frac{B}{rV}; \tag{138}$$

whence, we have

$$A = -\frac{V}{r}, \qquad B = \frac{V}{r}(r^2 - V^2). \tag{139}$$

Substituting these values in Eqs. (136), we obtain

$$\left.\begin{aligned} \frac{r\cos\lambda}{r^2 - a^2} &= \frac{V\cos l}{V^2 - a^2}, \\ \frac{r\cos\mu}{r^2 - b^2} &= \frac{V\cos m}{V^2 - b^2}, \\ \frac{r\cos\nu}{r^2 - c^2} &= \frac{V\cos n}{V^2 - c^2}, \\ \frac{1}{r^2 - V^2} &= V^2\left[\frac{\cos^2 l}{(V^2 - a^2)^2} + \frac{\cos^2 m}{(V^2 - b^2)^2} + \frac{\cos^2 n}{(V^2 - c^2)^2}\right] \\ &= r^2\left[\frac{\cos^2\lambda}{(r^2 - a^2)^2} + \frac{\cos^2\mu}{(r^2 - b^2)^2} + \frac{\cos^2\nu}{(r^2 - c^2)^2}\right]. \end{aligned}\right\} \tag{140}$$

The first three equations (140) can be placed under the form,

$$\left.\begin{aligned} \cos\lambda - \frac{V}{r}\cos l &= (r^2 - V^2)\frac{\cos\lambda}{r^2 - a^2}, \\ \cos\mu - \frac{V}{r}\cos m &= (r^2 - V^2)\frac{\cos\mu}{r^2 - b^2}, \\ \cos\nu - \frac{V}{r}\cos n &= (r^2 - V^2)\frac{\cos\nu}{r^2 - c^2}. \end{aligned}\right\} \tag{141}$$

Adding these, after multiplying them by cos λ, cos μ, cos ν, respectively, and reducing by Eq. (131), we have

$$1 - \frac{V^2}{r^2} = (r^2 - V^2)\left(\frac{\cos^2\lambda}{r^2 - a^2} + \frac{\cos^2\mu}{r^2 - b^2} + \frac{\cos^2\nu}{r^2 - c^2}\right); \qquad (142)$$

whence, dividing by $r^2 - V^2$, we have

$$\frac{\cos^2\lambda}{r^2 - a^2} + \frac{\cos^2\mu}{r^2 - b^2} + \frac{\cos^2\nu}{r^2 - c^2} = \frac{1}{r^2}, \qquad (143)$$

the polar equation of the wave surface.

108. A more advantageous form for discussion can be obtained by subtracting the identical equation,

$$\frac{1}{r^2} = \frac{\cos^2\lambda}{r^2} + \frac{\cos^2\mu}{r^2} + \frac{\cos^2\nu}{r^2}, \qquad (144)$$

from the equation above, by which there results

$$\frac{a^2\cos^2\lambda}{r^2 - a^2} + \frac{b^2\cos^2\mu}{r^2 - b^2} + \frac{c^2\cos^2\nu}{r^2 - c^2} = 0. \qquad (145)$$

109. To obtain the equation of the wave surface in rectangular co-ordinates substitute for $\cos\lambda$, $\cos\mu$, $\cos\nu$, and r, their equals, $\frac{x}{r}$, $\frac{y}{r}$, $\frac{z}{r}$, and $\sqrt{x^2 + y^2 + z^2}$; whence, we have

$$\left.\begin{aligned}(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 \\ - b^2(c^2 + a^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0.\end{aligned}\right\} \qquad (146)$$

This equation being of the fourth degree, the surface is of the fourth order, and, as will be shown hereafter, consists of two distinct nappes, having but four points in common.

110. If two of the velocities become equal, as, for example, $b = c$, the equation gives

$$x^2 + y^2 + z^2 = b^2, \qquad (147)$$

$$a^2x^2 + b^2(y^2 + z^2) = a^2b^2, \qquad (148)$$

which shows that the wave surface, under this supposition, consists of a spherical surface and that of an ellipsoid of revolution tangent to the sphere at the extremity of its polar axis.

Finally, if the three principal velocities become equal, or $a = b = c$, as in isotropic media, Eq. (146) becomes

$$x^2 + y^2 + z^2 = a^2, \tag{149}$$

and the wave surface becomes spherical, as has been heretofore shown.

111. ***Construction of the Wave Surface by Means of the Ellipsoid* (W).** Let us suppose that the ellipsoid (W),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{150}$$

be cut by any plane through its centre, and that distances be laid off on the normal equal to the semi-axes of the elliptical section. Referring to the construction of the double-napped surface of elasticity by means of the ellipsoid (E),

$$a^2x^2 + b^2y^2 + c^2z^2 = 1, \tag{151}$$

we see that in designating the polar co-ordinates of the points constructed by the aid of (W) by r, λ, μ, ν, the equation of their loci can be obtained from the equation of the double-napped surface,

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0, \tag{152}$$

by substituting for V^2, a^2, b^2, c^2, l, m, n, respectively, $\frac{1}{r^2}$, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, λ, μ, ν. We thus obtain

$$\frac{\cos^2 \lambda}{\frac{1}{r^2} - \frac{1}{a^2}} + \frac{\cos^2 \mu}{\frac{1}{r^2} - \frac{1}{b^2}} + \frac{\cos^2 \nu}{\frac{1}{r^2} - \frac{1}{c^2}} = 0, \tag{153}$$

or

$$\frac{a^2 \cos^2 \lambda}{r^2 - a^2} + \frac{b^2 \cos^2 \mu}{r^2 - b^2} + \frac{c^2 \cos^2 \nu}{r^2 - c^2} = 0, \tag{154}$$

which is the equation of the wave surface. Hence, points of the wave surface can be constructed from the ellipsoid (W) in precisely the same manner as points of the surface of elasticity from the ellipsoid (E), except that in the former, distances *equal* to the semi-axes are laid off on the normal, and in the latter the distances are *equal to the reciprocals* of the semi-axes.

112. *Direction of the Vibration at any Point of the Wave Surface.* Let us consider any plane wave tangent to the wave surface; the displacement propagated by this plane wave makes the angles α, β, γ, with the axes; the radius vector of the wave surface at the point of tangency makes with the axes the angles λ, μ, ν; therefore the angle between these two lines will determine the required direction.

Eliminate in Eq. (129) the angles l, m, n, by means of the first three of Eqs. (140), which, with the last of Eqs. (136), will give

$$\left.\begin{aligned} \frac{\dfrac{\cos\lambda}{r^2-a^2}}{\cos\alpha} &= \frac{\dfrac{\cos\mu}{r^2-b^2}}{\cos\beta} = \frac{\dfrac{\cos\nu}{r^2-c^2}}{\cos\gamma} \\ &= \sqrt{\frac{\cos^2\lambda}{(r^2-a^2)^2} + \frac{\cos^2\mu}{(r^2-b^2)^2} + \frac{\cos^2\nu}{(r^2-c^2)^2}} \\ &= \frac{V}{r}\sqrt{\frac{\cos^2 l}{(V^2-a^2)^2} + \frac{\cos^2 m}{(V^2-b^2)^2} + \frac{\cos^2 n}{(V^2-c^2)^2}} \\ &= \frac{V}{r}\sqrt{\frac{1}{BVr}} = \frac{1}{r\sqrt{r^2-V^2}}; \end{aligned}\right\} \quad (155)$$

whence,

$$\left.\begin{aligned} \frac{\cos\lambda}{r^2-a^2} &= \frac{\cos\alpha}{r\sqrt{r^2-V^2}}, \\ \frac{\cos\mu}{r^2-b^2} &= \frac{\cos\beta}{r\sqrt{r^2-V^2}}, \\ \frac{\cos\nu}{r^2-c^2} &= \frac{\cos\gamma}{r\sqrt{r^2-V^2}}, \end{aligned}\right\} \quad (156)$$

Substituting in Eq. (143) of the wave surface, we have

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = \sqrt{1-\frac{V^2}{r^2}}. \quad (157)$$

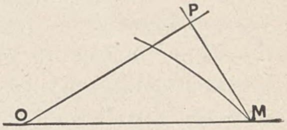

Figure 16.

In the figure, let M be any point of the wave surface, OM the radius vector, and OP the perpendicular V on the tangent plane to the wave surface; $\frac{V}{r}$ is

then the cosine of POM, and $\sqrt{1 - \frac{V^2}{r^2}}$ is its sine; hence, OMP is complementary to POM, and therefore the vibrations at M are directed along the line PM. We conclude, therefore, that the direction of the vibrations of the molecule at any point of the wave surface is along the projection of the radius vector on the tangent plane at that point.

When the tangent plane is normal to the radius vector, as is the case at the extremities of the axes, this determination is not applicable, but the direction is in these cases easily found. The plane OMP, which contains the radius vector and the direction of the corresponding vibration, is called the *plane of vibration.*

113. ***Relations between the Directions of Normal Propagation of Plane Waves, the Directions of Radii-Vectores of the Wave Surface, and the Directions of Vibrations.*** By the preceding theorem we have seen that, in any plane wave whatever, the normal to this plane, the direction of the vibrations in this wave, and the radius vector drawn to the wave surface at the point of tangency, are all contained in the same plane. Besides, for each normal direction of propagation of a plane wave, there correspond for the vibrations, two directions parallel to the axes of the elliptical section of the ellipsoid (E) made by a parallel plane. These directions, therefore, being perpendicular to each other, the planes which contain, at the same time, the same direction of normal propagation, the two vibrations, and the two corresponding radii-vectores, are rectangular.

114. Since the wave surface has two nappes, each radius vector will give two directions for the vibrations. We will now show that the planes which contain a radius vector and the directions of the two corresponding vibrations are also rectangular; and for this purpose we shall show that the two vibrations which correspond to the same radius vector are contained in the two planes passing through this radius vector and the axes of the elliptical section, that a plane perpendicular to the radius vector cuts out of the ellipsoid (W).

115. Let ϕ, ψ, χ, be the angles made by one of the axes of this elliptical section with the co-ordinate axes; it is then necessary to demonstrate that the three lines (α, β, γ), (λ, μ, ν), (ϕ, ψ, χ), are all in the same plane.

Eq. (129), which gives the relations existing between the angles α, β, γ, made by one of the axes of ellipsoid (E) with the co-ordinate axes, and the right line l, m, n, perpendicular to the elliptical section, can be applied to the analogous case of the elliptical section of (W) and the normal radius vector, by replacing in this equation V^2, α, β, γ, l, m, n, a^2, b^2, c^2, by $\frac{1}{r^2}$, ϕ, ψ, χ, λ, μ, ν, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, respectively, which will give

$$\left.\begin{aligned}\frac{\frac{a^2\cos\lambda}{r^2-a^2}}{\cos\phi}&=\frac{\frac{b^2\cos\mu}{r^2-b^2}}{\cos\psi}=\frac{\frac{c^2\cos\nu}{r^2-c^2}}{\cos\chi}\\ &=\sqrt{\frac{a^4\cos^2\lambda}{(r^2-a^2)^2}+\frac{b^4\cos^2\mu}{(r^2-b^2)^2}+\frac{c^4\cos^2\nu}{(r^2-c^2)^2}}.\end{aligned}\right\} \quad (158)$$

Let the auxiliary right line defined by the angles (A, B, C) be drawn perpendicular to the two right lines (λ, μ, ν) and (ϕ, ψ, χ); we will have

$$\left.\begin{aligned}\cos A\cos\lambda+\cos B\cos\mu+\cos C\cos\nu=0,\\ \cos A\cos\phi+\cos B\cos\psi+\cos C\cos\chi=0.\end{aligned}\right\} \quad (159)$$

Replacing, in the last equation, for $\cos\phi$, $\cos\psi$, $\cos\chi$, the quantities proportional to them in Eq. (158), we have

$$\frac{a^2\cos\lambda}{r^2-a^2}\cos A+\frac{b^2\cos\mu}{r^2-b^2}\cos B+\frac{c^2\cos\nu}{r^2-c^2}\cos C=0. \quad (160)$$

Adding this to the first of Eqs. (159), we have

$$\frac{\cos\lambda}{r^2-a^2}\cos A+\frac{\cos\mu}{r^2-b^2}\cos B+\frac{\cos\nu}{r^2-c^2}\cos C=0; \quad (161)$$

and recollecting that the relations

$$\frac{\frac{\cos\lambda}{r^2-a^2}}{\cos\alpha}=\frac{\frac{\cos\mu}{r^2-b^2}}{\cos\beta}=\frac{\frac{\cos\nu}{r^2-c^2}}{\cos\gamma} \quad (162)$$

exist, we have finally

$$\cos\alpha\cos A+\cos\beta\cos B+\cos\gamma\cos C=0. \quad (163)$$

Hence, the three right lines (α, β, γ), (λ, μ, ν), (ϕ, ψ, χ), being perpendicular to the right line (A, B, C), are all contained in the

same plane, and therefore we conclude that the planes which contain, at the same time, the same radius vector, the two vibrations, and the two corresponding directions of the normal propagation, are rectangular.

116. ***Discussion of the Wave Surface.*** Resuming Eq. (146),

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 \\ - b^2(c^2 + a^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0,$$

and making in succession $x = 0$, $y = 0$, $z = 0$, we get for the sections made by the co-ordinate planes yz, xz, and xy, respectively,

$$(y^2 + z^2 - a^2)(b^2y^2 + c^2z^2 - b^2c^2) = 0, \quad (164)$$

$$(x^2 + z^2 - b^2)(a^2x^2 + c^2z^2 - a^2c^2) = 0, \quad (165)$$

$$(x^2 + y^2 - c^2)(a^2x^2 + b^2y^2 - a^2b^2) = 0. \quad (166)$$

Remembering that $a > b > c$, we see that the section in the plane yz will be a circle whose radius is a, entirely outside of an ellipse whose semi-axes are b and c. The section in the plane xy will be a circle with radius c, entirely within the ellipse whose semi-axes are a and b. That in the plane xz will be a circle with radius

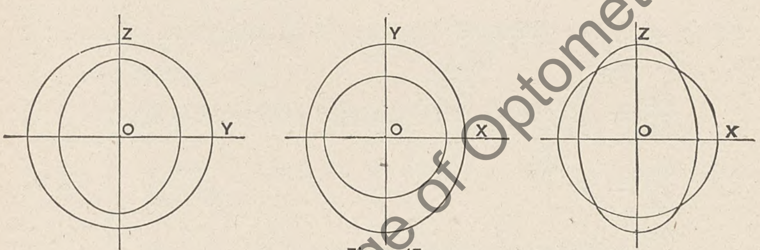

Figure 17.

b, intersecting at four points the ellipse whose semi-axes are a and c. The axis of x pierces the surface at distances equal to $\pm b$ and $\pm c$ from the centre, that of y at distances of $\pm a$ and $\pm c$, and that of z at $\pm a$ and $\pm b$.

117. The surface of elasticity of two nappes cuts the axes in the same points. These principal axes of elasticity have in turn represented the square roots of the elastic forces developed along the

three axes of elasticity, the principal velocities of wave propagation, the axes of the ellipsoids, and now serve to fix points on the surface of elasticity and on the wave surface.

118. Let Eq. (146) be represented by $L = 0$, and the angles which a tangent plane to the surface at any point makes with the co-ordinate planes xy, xz, yz, by A, B, C, respectively; then we will have

$$\cos A = \frac{1}{\omega} \cdot \frac{dL}{dz}, \quad \cos B = \frac{1}{\omega} \cdot \frac{dL}{dy}, \quad \cos C = \frac{1}{\omega} \cdot \frac{dL}{dx}, \tag{167}$$

in which,

$$\frac{1}{\omega} = \frac{1}{\sqrt{\left(\frac{dL}{dx}\right)^2 + \left(\frac{dL}{dy}\right)^2 + \left(\frac{dL}{dz}\right)^2}}. \tag{168}$$

Taking the differential coefficients of L with respect to x, y, z, we will have

$$\left.\begin{aligned} \frac{dL}{dx} &= 2x(a^2x^2 + b^2y^2 + c^2z^2) + 2a^2x(x^2+y^2+z^2-b^2-c^2), \\ \frac{dL}{dy} &= 2y(a^2x^2 + b^2y^2 + c^2z^2) + 2b^2y(x^2+y^2+z^2-a^2-c^2), \\ \frac{dL}{dz} &= 2z(a^2x^2 + b^2y^2 + c^2z^2) + 2c^2z(x^2+y^2+z^2-a^2-b^2). \end{aligned}\right\} \tag{169}$$

For $y = 0$, the point of tangency is in the plane xz, and we have

$$\left.\begin{aligned} \frac{dL}{dx} &= 2x(a^2x^2 + c^2z^2) + 2a^2x(x^2 + z^2 - b^2 - c^2), \\ \frac{dL}{dy} &= 0, \\ \frac{dL}{dz} &= 2z(a^2x^2 + c^2z^2) + 2c^2z(x^2 + z^2 - a^2 - b^2), \end{aligned}\right\} \tag{170}$$

the second equation showing that the tangent plane is normal to the plane xz.

For $y = 0$, the equation of the surface gives

$$x^2 + z^2 - b^2 = 0, \qquad a^2x^2 + c^2z^2 - a^2c^2 = 0; \tag{171}$$

whence, for the co-ordinates x and z, we have

$$x = \pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \qquad z = \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}, \tag{172}$$

which are real so long as $a > b > c$. There are then four points of intersection in the plane xz. Substituting these values in Eqs. (167), we obtain

$$\cos A = \frac{0}{0}, \qquad \cos B = \frac{0}{0}, \qquad \cos C = \frac{0}{0}. \tag{173}$$

119. The interpretation of these indeterminate values of the cosines is, that at the points considered, a tangent plane to the wave surface may have any position whatever with respect to the co-ordinate planes. This property shows that these points are the vertices of conoidal cusps, each having a tangent cone. These points, called *umbilics,* belong to the exterior and interior nappe of the wave surface, just as the vertex of a cone is common to its upper and lower nappes.

120. The equation of the right lines joining these points, OI, OI′, through the centre in the plane xz is

$$z = \pm \frac{a}{c}\sqrt{\frac{b^2 - c^2}{a^2 - b^2}}\,x, \tag{174}$$

which shows that the lines are normal to the circular sections of the ellipsoid (W). The lines themselves are called *axes of exterior conical refraction.*

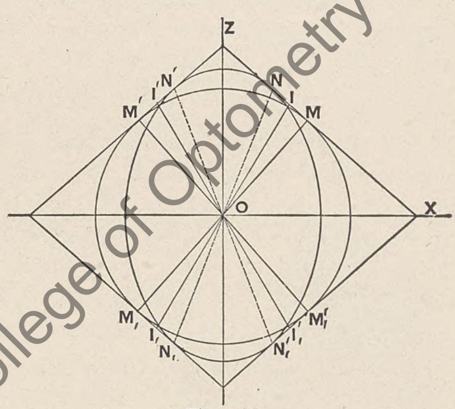

Figure 18.

121. If tangent lines be drawn to the ellipse and circle, as MN, M′N′, they will be parallel to each other, two and two, and symmetrically placed with respect to the axes OX and OZ. The equations of these lines can easily be shown to be

$$z = \pm\sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x \pm b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}, \tag{175}$$

and hence the equation of the line drawn from O perpendicular to the tangent to be

$$z = \mp \sqrt{\frac{b^2 - c^2}{a^2 - b^2}}\, x, \tag{176}$$

which shows that these lines are normal to the circular sections of the ellipsoid (E).

122. From the properties of this ellipsoid, we see that a plane wave perpendicular to one of the right lines MM_1, $M'M_1'$, and at the same time perpendicular to xz, can be propagated without alteration, whatever may be the direction of the displacement in its plane, and that the velocity of propagation of this plane wave is independent of the direction of the displacement. The lines MM_1, $M'M_1'$, are called the *optic axes* of the medium, or *axes of interior conical refraction.*

123. We see, by comparing Eqs. (174) and (176), that the lines OI and OM differ by the factor $\frac{a}{c}$ in their tangents. This ratio is always very nearly unity, and therefore the lines have nearly the same direction.

124. The planes drawn through the four tangents MN, M′N′, etc., perpendicular to the plane xz, are tangent to the wave surface along the circumferences of circles, which are projected in the lines MN, M′N′, etc. To show this, let

$$F(x, y, z) = 0 \tag{177}$$

be the equation of the wave surface; then, for points in the plane perpendicular to xz, we have

$$\frac{dF}{dy} = y\,(a^2x^2 + b^2y^2 + c^2z^2) + b^2y\,(x^2 + y^2 + z^2) - b^2(a^2 + c^2)\,y = 0. \tag{178}$$

which can be satisfied by placing

$$y = 0, \tag{179}$$

and $$(a^2 + b^2)\,x^2 + 2b^2y^2 + (b^2 + c^2)\,z^2 - b^2(a^2 + c^2) = 0. \tag{180}$$

The first of these equations gives the points of contact in the plane xz; the second represents an ellipsoid. If we combine the equation of the ellipsoid (180) with the equation of the wave surface, eliminating y^2, the resulting equation will be the projection on the plane xz of the intersections of these surfaces, and since the co-

ordinates of the points projected satisfy the condition, $\frac{dF}{dy} = 0$, all the points of the wave surface in the tangent plane which is perpendicular to xz, will be obtained from this intersection and projection. The resulting equation after reduction can be put in the form of

$$\left.\begin{array}{l} \left(z + \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x + b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times \left(z - \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x + b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times \left(z + \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x - b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \\ \times \left(z - \sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\,x - b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}\right) \end{array}\right\} = 0. \qquad (181)$$

This equation can be satisfied by placing each factor separately equal to zero, and each will then be the equation of a plane passed through one of the tangent lines MN, M′N′, M_1N_1, $M_1'N_1'$; hence, each of the four planes touch the surface in those points determined by its intersection with the ellipsoid, and it is readily seen that these curves are the circular sections of the ellipsoid, Eq. (180). The four planes

$$z = \pm x\sqrt{\frac{a^2 - b^2}{b^2 - c^2}} \pm b\sqrt{\frac{a^2 - c^2}{b^2 - c^2}} \qquad (182)$$

are called the *singular tangent planes* of the wave surface.

125. The circles are, in fact, the edges of the conoidal or umbilic cusps, determined by the surface of the tangent cones, reaching their limits by becoming planes in the gradual increase of the inclination of their elements, as the tangential circumference recedes from the cusp points.

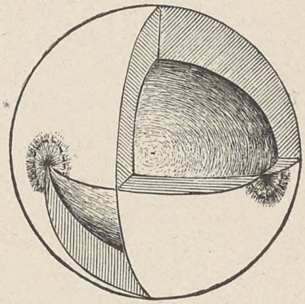

Figure 19.

It thus appears that the general wave surface consists of two nappes, the one wholly within the other, except at four points, where they unite.

Fig. 19 represents a model of the wave surface, with sections made by the co-ordinate planes, so as to show the interior nappe.

126. *Relations between the Velocities and Positions of Plane Waves with respect to the Optic Axes.* For each direction of normal propagation, two plane waves travel with different velocities, determined, as we have seen, by the equation of the surface of elasticity,

$$\frac{\cos^2 l}{V^2 - a^2} + \frac{\cos^2 m}{V^2 - b^2} + \frac{\cos^2 n}{V^2 - c^2} = 0.$$

This equation can be put under the form

$$\left.\begin{aligned} V^4 - [(b^2+c^2)\cos^2 l + (a^2+c^2)\cos^2 m + (a^2+b^2)\cos^2 n]\,V^2 \\ + b^2c^2\cos^2 l + a^2c^2\cos^2 m + a^2b^2\cos^2 n = 0\,; \end{aligned}\right\} \quad (183)$$

and representing the two square roots by V'^2 and V''^2, we have

$$\left.\begin{aligned} V'^2 + V''^2 = (b^2 + c^2)\cos^2 l + (a^2 + c^2)\cos^2 m \\ + (a^2 + b^2)\cos^2 n, \end{aligned}\right\} \quad (184)$$

$$V'^2 V''^2 = b^2c^2\cos^2 l + a^2c^2\cos^2 m + a^2b^2\cos^2 n. \quad (185)$$

Let θ', θ'', be the angles that the direction of normal propagation makes with the optic axes, and ϕ and $180° - \phi$ the angles that the optic axes make with the axis x, the axis of greatest elasticity, then we have

$$\cos\phi = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \qquad \sin\phi = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}, \quad (186)$$

$$\left.\begin{aligned} \cos\theta' &= \quad\cos\phi\cos l + \sin\phi\cos n, \\ \cos\theta'' &= -\cos\phi\cos l + \sin\phi\cos n; \end{aligned}\right\} \quad (187)$$

whence, we get

$$\cos l = \frac{\cos\theta' - \cos\theta''}{2\cos\phi} = \frac{\cos\theta' - \cos\theta''}{2}\sqrt{\frac{a^2 - c^2}{a^2 - b^2}}, \quad (188)$$

$$\cos n = \frac{\cos\theta' + \cos\theta''}{2\sin\phi} = \frac{\cos\theta' + \cos\theta''}{2}\sqrt{\frac{a^2 - c^2}{b^2 - c^2}}. \quad (189)$$

Substituting these values of $\cos l$ and $\cos n$ in Eqs. (184) and (185), and replacing $\cos^2 m$ by $1 - \cos^2 l - \cos^2 n$, we obtain

$$
\left.\begin{aligned}
V'^2 + V''^2 &= a^2 + c^2 - \frac{(\cos\theta' - \cos\theta'')^2}{4}(a^2 - c^2) \\
&\qquad + \frac{(\cos\theta' + \cos\theta'')^2}{4}(a^2 - c^2) \\
&= a^2 + c^2 + (a^2 - c^2)\cos\theta'\cos\theta'',
\end{aligned}\right\} \quad (190)
$$

$$
\left.\begin{aligned}
V'^2 V''^2 &= a^2c^2 - \frac{(a^2 - c^2)\,c^2}{4}(\cos\theta' - \cos\theta'')^2 \\
&\qquad + \frac{(a^2 - c^2)\,a^2}{4}(\cos\theta' + \cos\theta'')^2 \\
&= a^2c^2 + \frac{(a^2 - c^2)^2}{4}(\cos^2\theta' + \cos^2\theta'') \\
&\qquad + \frac{(a^2-c^2)\,(a^2+c^2)}{2}\cos\theta'\cos\theta''\,;
\end{aligned}\right\} \quad (191)
$$

whence,

$$
\left.\begin{aligned}
(V'^2 - V''^2)^2 &= (V'^2 + V''^2)^2 - 4V'^2V''^2 \\
&= (a^2 + c^2)^2 + (a^2-c^2)^2\cos^2\theta'\cos^2\theta'' - 4a^2c^2 \\
&\qquad - (a^2 - c^2)^2(\cos^2\theta' + \cos^2\theta'') \\
&= (a^2 - c^2)^2(1 - \cos^2\theta')(1 - \cos^2\theta'') \\
&= (a^2 - c^2)^2\sin^2\theta'\sin^2\theta''\,;
\end{aligned}\right\} \quad (192)
$$

and finally, $$V'^2 - V''^2 = (a^2 - c^2)\sin\theta'\sin\theta''. \quad (193)$$

This equation establishes the relation between the velocities of the two plane waves which belong to the same direction of normal propagation, and the angles that this direction makes with the optic axes.

127. The directions of the two vibratory motions can be determined by means of the optic axes. These directions are parallel to the axes of the elliptical section of (E) made by the plane normal to the direction of propagation; but the elliptical section is cut by the planes of the two circular sections of the ellipsoid in two equal diameters of the ellipse, since they are equal to the radius b of the circular section; they are therefore equally inclined to the axes of the ellipse. The optical axes being normal to the circular sections, are projected on the plane of the ellipse in two diameters which are

perpendicular to those just spoken of, and are therefore also equally inclined to the axes of the ellipse. But these projections are the traces of the planes containing the directions of the normal propagation and each optic axis. We therefore conclude, *that the bisecting planes of the diedral angle formed by the planes containing the direction of any normal propagation and each of the optic axes, are the planes of vibration of the two plane waves corresponding to this normal propagation.*

128. The plane xz being the plane of the optic axes, any direction of normal propagation in this plane will make the diedral angle 0° and 180°, and hence the planes of vibration will be the principal plane xz and a plane containing y and the direction of propagation.

129. ***Relations between the Velocities of Two Rays which are Coincident in Direction, and the Angles that this Direction makes with the Axes of Exterior Conical Refraction.*** The expressions

$$\pm\sqrt{\frac{a^2-b^2}{a^2-c^2}} \quad \text{and} \quad \sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

being the cosines of the angles that the optic axes make with the axes of x, z, and making use of the analogy existing between the ellipsoid (E) to the surface of elasticity, and the ellipsoid (W) to the wave surface, we will have, by substituting for a^2, b^2, c^2, in the above, $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, the expressions

$$\pm\frac{c}{b}\sqrt{\frac{a^2-b^2}{a^2-c^2}} \quad \text{and} \quad \frac{a}{b}\sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

for the cosines of the angles that the axes of exterior conical refraction make with the axes of x, z.

If then r and r', the two coincident radii-vectores of the wave surface, represent the ray velocities propagated in the same direction, and u' and u'' be the angles made by this direction with the two axes of exterior conical refraction, a discussion in every way analogous to that above for the optic axes will determine the required relation. This relation may be at once determined by replacing V', V'', θ', θ'', in Eq. (193), by $\frac{1}{r'}$, $\frac{1}{r''}$, u', u'', respectively; we then have

$$\frac{1}{r'^2} - \frac{1}{r''^2} = \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin u' \sin u''. \tag{194}$$

130. The axes of exterior conical refraction being normal to the circular sections of ellipsoid (W), by a similar course of reasoning as in Art. (127), we will arrive at the theorem, that *the bisecting planes of the diedral angle, formed by the planes containing any radius vector of the wave surface and each of the axes of exterior conical refraction, are the planes of vibration of the two rays corresponding to this radius vector.*

131. Thus, from the wave surface we can determine:

1°. The position of the refracted plane waves by its tangent planes.

2°. The direction of the two corresponding rays by the points of contact of the two parallel tangent planes.

3°. The velocities of the two rays by the lengths of the radii-vectores drawn to the points of contact.

4°. The velocities of the two plane waves by the normals from the centre upon the tangent planes.

5°. The interior directions of the molecular vibrations by the projection of the radii-vectores on the tangent planes.

6°. The plane of vibration by the plane of the normals and vibrations.

132. We have now shown that when any arbitrary displacement is made in any homogeneous medium, a disturbance is propagated in all directions from the origin, and that it is materially affected and controlled by the elastic forces developed. In accepting the conclusions which result, the limitations which have been primarily established must be kept in mind, to avoid the danger of accepting these results other than as exceptional and governed by the admitted hypotheses and by the accuracy of the mathematical processes employed. Observation and experiment are essential to ascertain to what extent the corresponding physical phenomena conform to these deductions. They are to be used, when at variance, to modify the hypotheses, and ultimately through this modification to approach nearer and nearer the true theories of the physical science.

133. The fundamental hypotheses upon which the foregoing discussion is in part based are as follows:

1°. The admission of such a constitution of the medium that

while it is variable around any molecule, it is similarly variable around all the molecules. The propagation of the disturbance without change of direction of the vibrations, when the latter are excited along the singular directions depends on this assumption. This inequality of elasticity is unquestionably exhibited in the phenomena of crystallization.

2°. That the excursions of the displaced molecules are so small that the resultant elastic forces in any direction are proportional to the displacement. This implies that the distances separating the adjacent molecules are very great in comparison with their displacements.

3°. The principle of the coexistence and superposition of small motions, by which any vibration can be replaced by others equivalent to it which are rectilineal.

4°. The inefficacy of the longitudinal component of the elastic force in light undulations, and the fact therefore of transversal vibrations. (The grounds of this assumption are to be given subsequently.)

5°. The correlation of the total intensity of the elastic force to certain velocities, and its identity with that expressed by the equation

$$V = \sqrt{\frac{e}{d}}.$$

6°. The principle of interference, by which the motion is entirely destroyed everywhere, except upon certain surfaces, which may be regarded as the loci of first arrival.

134. The agreement of the results obtained by experiment and from observation with the deductions from the theory is almost complete, while the crucial test of prediction in several noted instances leaves but little doubt of the truth of the undulatory theory. The utility of the determination of the wave surface and of its thorough discussion is thus happily verified, by its almost complete capability of satisfactorily explaining most, if not all, of the phenomena of physical optics. While in the limited course prescribed for the Academy we are unable to undertake the complete solution, we have, in the short and elementary discussion here presented, obtained sufficient data to prosecute the study of sound and light to the extent necessary for our purposes, and in this study we will have frequent occasion to refer to the foregoing analysis.

PART II.

ACOUSTICS.

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135. The investigations of physical science show that all sensation has its origin in the state of relative motion of the molecules of some medium with which the organ of sensation is in sensible contact. Each sensation has its peculiar organ, which, with its nerve system, receives and transmits molecular kinetic energy to the brain, where it is transformed into sensation. The motions of the molecules are, in general, vibrations, which are conveyed by undulations from the source of disturbance in all directions throughout the medium.

136. Acoustics is that branch of physical science which treats of *sound*. The sensation of sound usually arises from the communication of a vibratory motion of the tympanic membrane of the ear, due to the slight and rapid changes of the air pressure upon its exterior surface, the vibratory motion of the air being caused by the vibration of other bodies.

137. The ear consists essentially of two parts, one being in communication with the external atmosphere, the other with the brain.

The first consists of an irregularly formed tube, beginning at the orifice of the external ear and ending at the pharynx. Nearly midway, the tympanic membrane, or drum-skin, of the ear crosses this tube obliquely, separating the external portion, called the *meatus*, from the part immediately within, called the *tympanum*. That portion of the tube leading from the tympanum to the pharynx, or cavity behind the tonsils, is called the *Eustachian* tube. The orifice of this tube at the tympanum is generally closed; but the act of swallowing opens it, whereupon the air on both sides of the tympanic membrane becomes uniform in density. These three portions of the first part of the ear generally, however, contain air differing in density. In the meatus the air responds to all changes,

however slight and rapid, taking place in the external atmosphere; while the air in the tympanum and Eustachian tube is not so affected, unless communication with the external atmosphere be made as above described.

138. The other part, sometimes called the internal ear, is surrounded by bone, except in two places, called the *round* and *oval* windows. The cavity thus formed is called the *bony labyrinth.* The windows are closed by membranes which separate the tympanum on the one side from the fluid contained in the labyrinth on the other. Connecting the tympanic membrane with the oval window is a series of small bones, whose function appears to be to transfer the vibrations from the former to the latter. The labyrinth is filled with liquid, having suspended in it many membraneous bags, also filled with liquid. Upon the surface of these bags are spread the terminal fibres of the auditory nerves, which, by special arrangements, are enabled to take up the energy communicated to the liquid in the labyrinth. The membrane of the round window readily yields to the pressure of the liquid, moving out and in as the oval window is moved in and out by the transfer of motion through the bones of the ear.

Thus the energy communicated to the air in the external ear is conveyed from the tympanic membrane, through the series of small bones in the tympanum, to the membrane of the oval window, thence to the liquid of the labyrinth, and finally to the auditory nerves. How this energy is transformed into sensation is unknown.

139. To represent, graphically, the variations of air pressure, we will make use of the *curve of pressure*, in which the abscissas correspond to the *times* and the ordinates to the *excess* of the pressure above its mean or average value. The pressure of the air, at any point, is assumed to be measured by the pressure of air of the same density and temperature upon a unit of area. Then take

$$y = f(t) \tag{195}$$

to represent any curve of pressure as

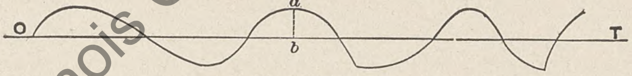

Figure 20.

in which $y = 0$ represents the standard or mean pressure, and

$y = \pm p$, a pressure above or below the standard pressure. Whenever the pressures are strictly proportional to the corresponding densities, as by the law of Mariotte, the same curve may also represent the curve of density. If we now assume that a curve similar to the above represents the slight and rapid changes of pressure of the air in contact with the tympanic membrane while the sensation of a particular sound exists, we see that these changes do not in general affect the average pressure of the air, for the areas above and below the axis of the curve are equal. A curve is said to be *periodic* when it consists of equal and like parts continuously repeated. The *wave length* of a periodic curve is the projection upon the axis of the smallest repeated portion.

140. The ear clearly distinguishes between a musical sound and a noise. The former is a uniform and sustained sensation, unaccompanied by any marked alteration, save that of intensity; while the latter is more or less varied and ununiform. When a sonorous body is sounding, the most ordinary examination is sufficient to show that it is in a state of vibration. The vibrations or oscillations of its parts set in corresponding motion the adjacent air-particles, which in turn transmit similar motions to the next following particles, and so on. The air, then, is ever passing through alternate states of condensation and rarefaction. When these vibrations are regular, periodic, and sufficiently rapid, the resulting sound is uniform in character and is called a *musical tone*. If the resulting sound arises from vibrations which are non-periodic, it is called a *noise*. Ordinary observation shows that few, if any, noises are perfectly unmusical; and few, if any, sounds are absolutely unmixed with noise.

141. ***Propagation of a Disturbance in an Indefinite Cylinder.*** Let us suppose the indefinite cylinder MN filled with air, and at the origin a piston p, capable of rapid to-and-fro motion. In the first place, let the piston be moved a distance ds from p to p', in the time dt. If the air were incompressible, it would be moved bodily over the distance ds. But being compressible, the air yields to the motion of the piston, and at the end of the time dt the compression will have reached a posi-

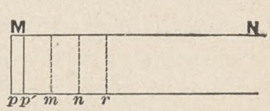

Figure 21.

tion m, so that the stratum of air, being condensed from pm to $p'm$, will exert an elastic force in excess of that due to its normal state. Call this excess δ. The increased elasticity of $p'm$ will cause it to expand in the only direction possible, towards the next stratum mn, which in turn becomes compressed. This second stratum reacts in both directions; on the side towards mp' it brings the molecules of mp' to rest, their acquired velocity having a tendency to cause them to pass beyond their positions of equilibrium; and on the side nr it compresses the next stratum, increasing its elasticity ultimately by δ. In this manner the compression is transmitted from stratum to stratum, throughout the whole length of the cylinder.

142. Let V be the velocity of propagation of the condensation, v the velocity of the piston; then we have

$$pp' = ds = vdt, \qquad pm = ds' = Vdt,$$

$$pm - pp' = p'm = (V - v)\,dt.$$

Supposing Mariotte's law applicable, and P to represent the normal pressure, we have

$$P : P + \delta :: p'm : pm :: (V - v)\,dt : Vdt;$$

or

$$\delta = P\frac{v}{V - v}. \tag{196}$$

Let p' now return to its primitive position p in the next successive dt. The first layer of the stratum will be dilated, occupying the new space $p'p$, and its pressure P will become $P - \delta$. The elastic force of the next layer P will become, by its expansion to the left, $P - \frac{\delta}{2}$, increasing that of the first also to $P - \frac{\delta}{2}$. But the velocity acquired by the molecules of the second layer will cause them to pass beyond their positions of equilibrium, so that its elastic force will diminish until it becomes $P - \delta$, at the instant the elastic force of the first layer, continually increasing, becomes P, its normal value. The third layer will, in turn, act on the second as the second has acted on the first, so that the dilatation corresponding to δ will travel the distance pm in the time dt, during which the piston is retracing its path $p'p$. The magnitude of δ will evidently depend on the value pp' and the time dt. If dt be constant and δ be varied, the condensations will vary with δ. The

analysis shows that the compressions and dilatations are propagated with equal velocities, and that these velocities are independent of the degree of condensation or of rarefaction, when the medium is the same and the amplitude is very small.

143. Let the prong of a tuning-fork $p \ldots . p'$ (Fig. 22) be displaced a very small but finite distance from its neutral position a. By its elasticity it will vibrate with equal displacements on each side of its position of equilibrium. Its velocity increases from zero at p to a maximum at a, and decreases in an exactly reverse manner to zero from a to p'. Let the duration of its motion from p to p' be divided into equal parts, each represented by dt, the epoch corresponding to the position p. From p the prong describes unequal but increasing distances during the successive dt's to the position a, and unequal but decreasing distances from a to p'. Each corresponding compression can be found from Eq. (196) by the substitution of the proper value of v, and these compressions or condensations will be propagated with a constant velocity V. While the prong is returning from p' to p, the rarefactions will increase from p' to a, and decrease from a to p, and their values may be determined from the same equation. The condensations will be symmetrically distributed with reference to the maximum condensation, neglecting the very small amplitude vp'. Likewise the rarefactions will be symmetrically distributed with respect to the maximum rarefaction.

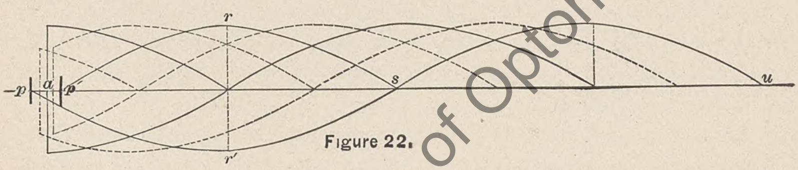

Figure 22.

144. The positive ordinates of the curve $p'rs$ represent the successive condensations, s being the position of the layer reached by the first condensation when the prong has arrived at p'; and the negative ordinates $pr's$ will represent the successive rarefactions when the first condensation has reached the position u, and the prong has returned to its primitive position p. The ordinates of the other curves represent either condensations or rarefactions, as indicated in the figure corresponding to the particular state and position of the prong.

145. In the figure, *pu,* the length of the wave is the distance traveled by the disturbance while the prong is making a complete vibration, and hence we have, n being the vibrational number and τ the periodic time,

$$\lambda = \frac{V}{n} = V\tau. \tag{197}$$

146. The mean velocity of the air molecules is evidently the same as that of the vibrating prong, and therefore this will vary with the vibrating body. In the example given, the mean velocity of the molecules is 2 mm. $\times 256 = 0.512$ m. The actual velocity of the air molecules continually varies, and at any time is proportional to the ordinates of the curve a quarter of a wave length in advance of the molecules considered. When the vibrating body has simple harmonic motion, the molecular velocity is given by

$$v = a \cos 2\pi \frac{t}{\tau}. \tag{198}$$

147. The value of a, the amplitude of the vibration, diminishes (Art. 72) according to the law of the inverse distance from the centre of disturbance; and for each value of a taken as constant within the wave length we have, by the above equation, sensibly exact values for the molecular velocity at any time.

148. When the vibrations of the body are sufficiently frequent during the unit of time, and of sufficient amplitude, the sensation of sound arises in the ear, which, however, we unconsciously refer to the vibrating body. A sonorous wave comprises the series of condensations and rarefactions arising from one complete vibration of the sounding body.

149. The sum of all the condensations in the condensed portion of the wave is represented by the area of the curve $p'rs$, and if it be divided by the duration of half the vibration, the mean condensation will result. Thus, take the amplitude of the oscillation of the tuning-fork, making 128 vibrations per second to be 1 mm., and the velocity of propagation to be 340 m.; then, from Eq. (196), we will have

$$\delta = P\frac{v}{V - v} = P\frac{\frac{1}{\frac{1}{256}}}{340000 - \frac{1}{\frac{1}{256}}} = P\frac{256}{340000 - 256} = P\frac{1}{1327}. \tag{199}$$

Hence, the change of density in the air, measured by the barometric height due to the mean condensation, is not greater than that due to 0.0226 inches of mercury, when a sound corresponding to 128 vibrations per second, and caused by the fork under the supposed conditions, is passing.

150. From the preceding discussion we see that we can neglect, in general, the absolute displacements of the air molecules, and consider the change in pressure and density as being alone propagated. Therefore, a file of elastic balls transmitting motion practically illustrates the state or condition of a series of air molecules during the propagation of a sonorous wave. An excellent illustration is also given by means of a chain cord. If it be attached at one of its extremities to a fixed point, and be held stretched at the other, the successive rings or spirals will assume positions of stable equilibrium with respect to each other, determined by the tension. These rings, for the purpose of illustration, may be taken to represent the contiguous air strata or particles in an indefinite tube, or upon any line along which sound is supposed to be propagated. If any ring be plucked, it will, when released, oscillate about its place of rest while the disturbance is being propagated in both directions to the points of support. Upon reaching these points the disturbance will be divided, a part proceeding in the new medium, and the remainder, being reflected, will retrace its path, to be again subdivided at the other end. This will continue until the whole energy of the original disturbance has been dissipated. By increasing the tension the disturbance will be more quickly propagated, and conversely. Now suppose, from the point of plucking, lines be drawn in all directions, and the same phenomena occur on these, then the behavior of each ring and the progressive motion of the disturbance illustrates what takes place in air during the passage of a sound wave along every right line drawn from the origin of the sounding body. In an isotropic and homogeneous medium, the disturbance moves with constant velocity, and the volume whose surface bounds the disturbed particles at any instant is a sphere whose radius is Vt.

151. The general properties of any sound are *intensity, pitch,* and *quality*.

Intensity is that property by which we distinguish the relative loudness of two tones of the same pitch and quality. We can also,

in general, determine which of two tones of different pitch and quality has the greater intensity. The air particles have but small displacements from their positions of relative rest, when the displacement is caused by the passage of a sound wave. The forces which urge them back to their positions of rest are assumed to vary directly with the degree of displacement. In Analytical Mechanics, it is shown that the periodic time of the air particle depends only upon its mass and the intensity of the force of restitution; and therefore, in the same medium, with given pressure, density, and temperature, for the same exciting cause, the periodic time will be constant, but the mean velocity of the air particle will vary with the size of the orbit. The kinetic energy in the moving particle, varying as the square of the velocity, will therefore, for the same exciting cause and the same medium, under the same circumstances of pressure, density, and temperature, vary directly as the square of the maximum displacement. By the law of the decay of energy, the intensity of the sound will therefore vary inversely as the square of the distance from the origin of the exciting cause. (Art. 72.)

152. *Pitch* is that property by which we distinguish the position of two tones in the musical scale, and thereby recognize which is the more acute and which the more grave. The pitch depends upon the frequency of the vibration; the greater the number of vibrations produced by a sounding body in a given time, the more acute will be the resulting sound. The siren is an instrument used to illustrate this fact. It consists essentially of a disk pierced with a number of equidistant holes, through which air is forced when it is put in rapid rotation. As the rotation increases, the sound gradually rises in pitch, and as it diminishes the pitch falls correspondingly. If a coin with a milled edge, or a cogged wheel, be put in rotation, and a card be held against it, the same changes in pitch will be observed. In these cases the single puff, or stroke of the card against the coin, or wheel, is essentially a noise, and when these strokes are multiplied sufficiently in a given time, the resulting effect is a note of definite pitch. So that a clearer distinction than that heretofore given should be made between a noise and a musical tone. To this distinction we will again refer.

153. The *quality* of a musical tone is that property by which we can distinguish whether two sounds of the same pitch, of either equal or unequal intensities, arise from the same or different sono-

rous bodies. This property enables us, within certain limits, to distinguish voices and the various sounds peculiar to different musical instruments. We have as yet only exacted that a musical sound shall be periodic and regular ; that is, that during any vibration the successive states of motion of the particle shall recur in the same order as in each of the previous vibrations. But it is evident that we may have an infinite variety of periodic motion, and it will be shown that the quality of the sound will vary with each variation of the periodic motion, the wave length remaining constant.

154. Every one has experienced the fact that more than one sound can be heard at once. Our attention can be, for the moment, fixed upon any one of the many sounds that are constantly occurring, and at the same time we may be conscious of the existence of the others. Therefore, the meeting of sound waves in the external ear does not, in general, result in mutual destruction, or in essential modification; while, at the same time, we must acknowledge that the air in contact with the tympanic membrane, at any given instant, can possess but one determinate pressure and density. The changes in pressure and density due to many exciting causes must, then, result from the superposition and coexistence of those arising from each separate cause, and, in general, without destruction or modification. We have here the application of the principle enunciated in Art. 204, Mechanics.

The more general statement of the law of the composition of displacements would be that demonstrated in the principle of the parallelogram of forces, but when the displacements are infinitely small, we can take, rigorously, the resultant displacement to be the algebraic sum of the component displacements. The diameter of the meatus at the tympanic membrane does not exceed 0.25 inch, and therefore, for sounds whose sources are at ordinary distances, the wave fronts at the position of the tympanic membrane coincide sensibly with their tangent planes, and the changes of density and pressure may be compounded by the law of small motions, without appreciable error.

155. Let the broken line, in the following diagram, represent the changes of pressure upon the tympanic membrane while a continuous noise, in which the ear recognizes no definite pitch, is sounding for a small part of a second, and let the dotted line represent another noise of the same duration.

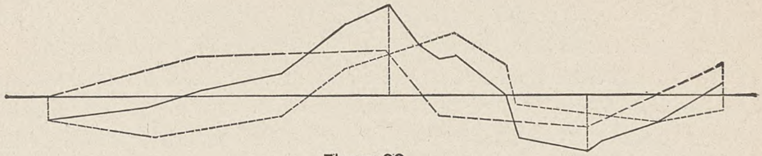

Figure 23.

Then, if both noises sound together, the resultant variation of pressure will be represented by the full line obtained by joining the extremities of the ordinates found by taking the algebraic sum of the ordinates of the separate curves.

These two noises do not, in general, unite into one, but are heard distinctly and simultaneously, except in the case where the two sounds are nearly alike, and the two curves nearly similar. Again, there is nothing in the resultant curve to suggest to the eye the nature of the two component curves. Hence, the ear possesses the property of separation; while the eye, according to this method of combination and representation, does not.

156. Let the component curves be periodic, two periods of O′X′ being equal to three of O″X″.

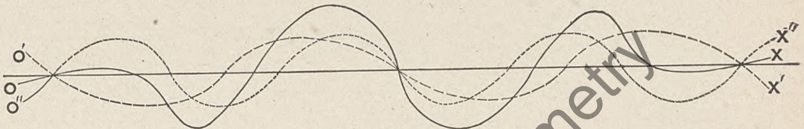

Figure 24.

The resultant curve OX will be a periodic curve, whose repeated portions are represented above. An examination of this curve by the eye gives no clue to its components, and we may resolve it into an indefinite number of pairs of components, but one of which would represent the two notes which sounding together will give us the resulting effect upon the ear. But if the ear resolves the composite note represented by OX, it must resolve, in like manner, O′X′ and O″X″. Observation confirms this deduction.

157. The only note the ear is incapable of resolving is that of the simple musical tone, and this incapability arises from the fact that such a tone is in reality perfectly simple, and not compound. The tones which are ordinarily called simple, are, in reality, compounded of a series of simple tones theoretically unlimited in number. Very few of them have sufficient intensity to be heard; but

these few form a combined note which is always the same under the same circumstances, and we habitually associate them together, and perceive them as a single note of a special character. But it is possible, with certain appliances, to partially analyze the composite note by an attentive study of the separate constituents.

Whenever two sounding bodies give notes whose tones form *consonant* combinations with each other, the difficulty of analysis is increased; when the combinations are *dissonant*, the analysis is less difficult.

158. A noise may therefore be defined to be a combination of musical tones, too near in pitch to be separately distinguished by the unassisted ear, or to be a combination of noises, each of which is made up of sounds so near each other in pitch as to be undistinguishable; the separate noises may be near or far apart in pitch. It is so complex, that its analysis is beyond the power of the unassisted ear. A simple musical tone, on the contrary, is incapable of resolution by reason of its absolute simplicity. Hence, strictly speaking, only simple tones have pitch. A simple musical tone will have a single determinate pitch. The pitch of a musical note must then be taken to mean the pitch of the *gravest* simple tone in its combination. If the higher simple tones be successively stopped out, the pitch, as defined, will remain unaltered, but the quality of the note will undergo variations until the single musical simple tone corresponding to the gravest tone is reached, beyond which no further modification can take place.

159. We will hereafter assume, as the fundamental simple tone, that component of any note which corresponds to the regular periodic curve of the given pitch. This distinction is important; for it is evident that there may be many periodic curves of the same pitch, and each may correspond to musical notes differing in quality.

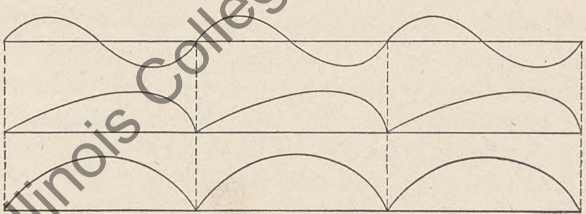

Figure 25.

The preceding curves (Fig. 25) represent notes of the same pitch, but of different quality. Helmholtz has shown that while every different quality of tone requires a different form of vibration, the converse is not necessarily true; *i. e.*, that different forms of vibration may not correspond to the same quality.

160. We have seen, page 45, that any physical condition, such as density, pressure, velocity, etc., which is measurable in magnitude or intensity, and which varies periodically with the time, may, by Eq. (195), be expressed as a function of the time. Hence, every periodic disturbance of the air, and particularly such disturbances as excite the sensation of a musical tone, can be resolved into its harmonic vibrations.

A single simple tone being represented by the simple harmonic curve

$$y' = a' \sin\left(\frac{2\pi x}{\lambda} + \alpha'\right), \tag{200}$$

and another of half wave length by

$$y'' = a'' \sin\left(\frac{2\pi x}{\frac{\lambda}{2}} + \alpha''\right) \tag{201}$$

the resultant curve will be represented by

$$y = a' \sin\left(\frac{2\pi x}{\lambda} + \alpha'\right) + a'' \sin\left(\frac{4\pi x}{\lambda} + \alpha''\right), \tag{202}$$

which has the same wave length, but a different amplitude and phase. This change in the amplitude and phase may be varied at pleasure, by conceiving the second curve to be shifted along the axis any distance from zero to λ, and again to pass through all values of the amplitude between any two limits. The resultant curve, in all cases, will, however, be a periodic curve of constant wave length.

161. Considering the simple musical tones which they represent then to be sounded together, with the same modifications, it has been found that the ear can distinguish the components when the attention is cultivated and directed to this effect. With a variation in phase only, the effect on the ear is constant and invariable, and hence we see that many different resultant curves may represent

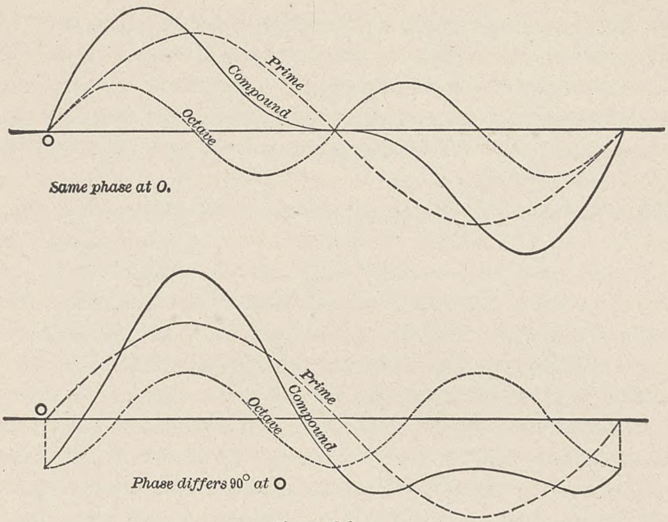

Figure 26.

essentially the same sensation. Thus, the two curves above, represent the same compound tone made up of the two simple tones, although the forms of the curves are quite different. The resultant tones are the same, both in quality and in pitch, but differ in intensity. By combining in the same way other simple tones of one-third, one-fourth the wave length, and so on, the quality will be changed, without affecting the pitch, as can be seen from the graphical construction, and heard by audible experience. In all these cases the untrained ear, by the aid of certain appliances, can always analyze the resultant sound into its component simple tones, and when trained, often without this assistance. When but one simple vibration of sufficient frequency and intensity to produce sensation alone exists, no such analysis takes place.

162. The investigations of Helmholtz have shown that the ear possesses the property of analysis of a single musical tone into its simple musical tones, each of which is distinctive in character, but which blend harmoniously into the single tone when sounded together. The wave lengths of these components are aliquot parts of the wave length of the fundamental, and the simple tones are called the *upper partials* of the fundamental or prime tone. Hence, from Art. 64 and these facts, we conclude that, when several sounding

bodies simultaneously excite different sounds, the variations of air density and the resultant displacements and velocities of the air particles in contact with the tympanic membrane are each equal to the algebraic sum of the corresponding changes of density, the displacements and the velocities which each system of waves would have separately produced, had it acted alone.

163. This analysis by the ear clearly shows, then, that the separate effects of the simple vibrations are, in general, neither modified nor destroyed, but actually exist, and it remains to be proved that such is really the case, independent of the peculiar sensation which is the result of their action upon the ear. Since Fourier's Theorem mathematically demonstrates that any form of vibration, no matter how varied its shape, can be expressed as the sum of a series of simple vibrations, its analysis into these simple vibrations is independent of the capacity of the eye to perceive by examining its representative curve whether it contains the simple harmonic curves or not, and if it does, what they are. All that the curve indicates is that the more regular its form, the greater the effect of its deeper or graver tones in comparison with its upper partials. Before proceeding to show that these component vibrations actually exist together, and that each can affect the ear or other sensitive vibrating body, let us now establish clearly the definitions pertaining to the subject.

164. A *simple* or *pendular* vibration is that which corresponds to the complete oscillation of a simple pendulum, and is graphically represented by the simple harmonic curve.

A *simple musical tone* is that effect produced upon the ear when a sonorous body is executing simple vibrations only, of sufficient frequency and amplitude to be heard. According to this definition, simple tones do not in reality exist ; but in the vibrations of such bodies as tuning-forks, the component vibrations which simultaneously exist with that of the gravest period, are generally nonperiodic with it, and so deficient in intensity that their influence is negligible, and we may regard such bodies as producing simple vibrations alone without sensible error.

165. A *single* musical tone may be either simple or compound. When compound, it is made up of its fundamental simple tone, together with its upper partial simple tones, each of which has a frequency of either twice, three times, or so on, that of its funda-

mental. It is due to the vibration of a single sonorous body which, during its motion, vibrates as a whole, and divides also into parts which vibrate twice, three times, and so on, as rapidly as the whole. One or more of these upper partials may be wanting during the vibration; when this occurs, the quality of the single musical tone is correspondingly affected.

A *composite* musical tone is composed of two or more single musical tones.

166. ***Musical Intervals.*** The extreme range of the human ear lies between 20 and 40000 simple vibrations per second. The corresponding wave lengths are obtained by dividing the velocity of sound by these numbers, and are approximately 54.6 feet and 0.0273 feet respectively, assuming the velocity of sound to be 1092 feet at 0° C. The ordinary sounds heard by the ear have a much less range; their vibrational numbers lie between 40 and 4000, corresponding to wave lengths of about 27.3 feet and 0.273 feet, respectively. When a stretched wire is put into vibration, and the tension continuously undergoes variation, the pitch of the sound passes by continuity from lower to higher, or the reverse, and we therefore experience the sensation of a musical interval between any two limiting tones. We may, then, define a musical interval by the ratio of the vibrational numbers of the two limiting tones. Thus, if the two tones correspond to the vibrational numbers 256 and 384, the name of the interval is the *fifth*, and it is expressed by the fraction $\frac{3}{2}$. Considering the simpler ratios that lie between two tones whose vibrational numbers are as 1 : 2, we obtain the following musical intervals:

Consonant.			*Dissonant.*		
Unison, . . .	1 : 1	$= \frac{1}{1}$	Major second, . . .	9 : 8	$= \frac{9}{8}$
Minor third, . .	6 : 5	$= \frac{6}{5}$	Minor second, . . .	10 : 9	$= \frac{10}{9}$
Major third, . .	5 : 4	$= \frac{5}{4}$	One-half major tone,	16 : 15	$= \frac{16}{15}$
Fourth, ...	4 : 3	$= \frac{4}{3}$	One-half minor tone,	25 : 24	$= \frac{25}{24}$
Fifth, ...	3 : 2	$= \frac{3}{2}$	Comma, ...	81 : 80	$= \frac{81}{80}$
Major sixth, . .	5 : 3	$= \frac{5}{3}$			
Octave, ...	2 : 1	$= \frac{2}{1}$			

The first are called *consonants*, because the effect is pleasing to

the ear when the tones of either of these intervals are sounded together. All other intervals within range of the octave are called *dissonants.*

167. The measure of the musical interval represented by the ratio $\frac{p}{q}$ is the $\log\frac{p}{q}$. This arises from the fact that if we consider any three tones whose vibrational numbers are p, q, and r, the musical interval between p and r must be equal to the sum of the two intervals between p and q, and q and r. If the ratios of the vibrational numbers were taken to measure the intervals, we would have, for the same interval, the expressions

$$\frac{r}{p} \quad \text{and} \quad \frac{q}{p}+\frac{r}{q}.$$

which are not equal to each other. But since

$$\frac{r}{p}=\frac{q}{p}\times\frac{r}{q}, \tag{203}$$

we have

$$\log\frac{r}{p}=\log\frac{q}{p}+\log\frac{r}{q}, \tag{204}$$

and we may therefore take the logarithm of the ratio of the vibrational numbers as the measure of the musical interval. The *name* of any interval, then, is the ratio of the vibrational numbers, and its *measure* is the logarithm of that ratio. The logarithms are usually taken in the common system.

168. *Musical Scales.* A series of tones at finite intervals is called a musical scale. If the vibrational numbers are in the proportion of the natural numbers, the musical scale is called the *harmonic* scale. When two tones whose interval is that of an octave are sounded together, we are conscious of a certain *sameness* of sensation, which is absent in all other intervals except multiples of the octave. We may then assume this interval as a natural unit, since it gives a periodic character to the scale. Whatever properties are found with regard to the tones in any octave, occur in the other octaves of a higher or lower pitch. The vibrational numbers of the tones of the harmonic scale, starting with a fundamental tone whose vibrational number is 128, will be as follows:

128 : 256 : 384 : 512 : 640 : 768 : 896 : 1024 : 1152 : 1280 : etc.

169. Examining these numbers, we see that each interval in any octave is divided, in the succeeding octave, into two intervals which can be obtained from the equation

$$\frac{n+1}{n} = \frac{2n+1}{2n} \times \frac{2n+2}{2n+1}, \tag{205}$$

n being the natural number which marks the position of the first tone of the lower interval in the harmonic scale. Thus we see that the interval 128 : 256, or the octave, is divided in the next octave into two intervals represented by $\frac{2n+1}{2n} = \frac{3}{2} = \frac{384}{256}$ and $\frac{2n+2}{2n+1} = \frac{4}{3} = \frac{512}{384}$. The first interval, 256 : 384, in the second octave is divided into the two intervals corresponding to $\frac{2n+1}{2n} = \frac{5}{4}$ and $\frac{2n+2}{2n+1} = \frac{6}{5}$ in the third octave; the second interval, 384 : 512, in the same octave, is in like manner divided into $\frac{2n+1}{2n} = \frac{7}{6}$ and $\frac{2n+2}{2n+1} = \frac{8}{7}$ in the third. The first interval, 512 : 640, in the third octave, is subdivided in the fourth octave into $\frac{9}{8}$ and $\frac{10}{9}$, and so on. Arranging all the intervals, with their corresponding subdivisions in the next higher octave, we have

1st octave, 128 : 256, interval $\frac{2}{1}$, subdivided in 2d octave into

$$256 : 364 = \frac{3}{2} \quad \text{and} \quad 384 : 512 = \frac{4}{3};$$

2d octave, 256 : 384, interval $\frac{3}{2}$, subdivided in 3d octave into

$$512 : 640 = \frac{5}{4} \quad \text{and} \quad 640 : 768 = \frac{6}{5};$$

2d octave, 384 : 512, interval $\frac{4}{3}$, subdivided in 3d octave into

$$768 : 896 = \frac{7}{6} \quad \text{and} \quad 896 : 1024 = \frac{8}{7};$$

3d octave, 512 : 640, interval $\frac{5}{4}$, subdivided in 4th octave into

$$1024 : 1152 = \frac{9}{8} \quad \text{and} \quad 1152 : 1280 = \frac{10}{9};$$

3d octave, 640 : 768, interval $\frac{6}{5}$, subdivided in 4th octave into

$$1280 : 1408 = \frac{11}{10} \quad \text{and} \quad 1408 : 1536 = \frac{12}{11}.$$

Thus every interval in the harmonic scale is divisible into two other intervals, whose ratios are those of consecutive numbers in the next higher octave.

170. *Perfect Accords.* A *perfect accord* is a series of three tones, called a *chord*, which, sounded simultaneously, give a particularly pleasing sensation to the ear. The *perfect major accord* consists of the three tones called the *tonic*, the *middle*, and the *dominant*, whose intervals are a major third and a fifth, or $\frac{5}{4}$ and $\frac{3}{2}$.

The *perfect minor accord* is composed of a minor third, $\frac{6}{5}$, and a fifth, $\frac{3}{2}$.

171. *The Diatonic Scale.* The tones of this scale are usually designated by letters or symbols, as follows :

C : D : E : F : G : A : B : *c* : *d* : etc.

ut or *do* : *re* : *mi* : *fa* : *sol* : *la* : *si* : *do* : *re* : etc.

Forming the perfect major accord on C as a tonic, we will have

$$\text{C} : \text{E} : \text{G},$$

$$1 : \frac{5}{4} : \frac{3}{2}.$$

Forming similar chords with C and G, by making C a dominant and G a tonic, we will have

$$F_1 : A_1 : C, \qquad G : B : d,$$

$$\frac{2}{3} : \frac{5}{6} : 1; \qquad \frac{3}{2} : \frac{15}{8} : \frac{9}{4}.$$

Arranging these three chords in order of their pitch, we find

$$F_1 : A_1 : C : E : G : B : d,$$

$$\frac{2}{3} : \frac{5}{6} : 1 : \frac{5}{4} : \frac{3}{2} : \frac{15}{8} : \frac{9}{4},$$

which is a musical scale of seven notes, rising one above another by alternate major and minor thirds.

Replacing in this scale F_1, A_1, by their higher octaves, and d by its lower octave, which is permissible, and arranging in order, we have

$$C : D : E : F : G : A : B : c,$$

$$1 : \frac{9}{8} : \frac{5}{4} : \frac{4}{3} : \frac{3}{2} : \frac{5}{3} : \frac{15}{8} : 2,$$

which is known as the diatonic scale. The names of the intervals heretofore used are now seen to come from the position of the notes in this scale with reference to the tonic; thus, the interval $\frac{9}{8}$ is a major second, the interval $\frac{5}{4}$ a major third, $\frac{4}{3}$ a fourth, $\frac{3}{2}$ a fifth, and so on. The first tone in the scale is called the *tonic*, the fifth the *dominant*, and the fourth the *subdominant*. Taking the vibrational number of the tonic C to be 24, we have the corresponding vibrational numbers of the diatonic scale,

$$C : D : E : F : G : A : B : c,$$

$$24 : 27 : 30 : 32 : 36 : 40 : 45 : 48.$$

172. The vibrational numbers of the other octaves are obtained from these by constantly doubling or halving them, according as we ascend or descend, the letters being properly accented to indicate in which octave the series is taken. Theoretically, the tones of the diatonic scale above belong to the harmonic scale, whose fundamental tone has *one* vibration per second. This fundamental

tone is *five* octaves below the subdominant; for $\frac{32}{1} = \left(\frac{2}{1}\right)^5$. We will hereafter take the octave whose tonic corresponds to 256 vibrations for that of comparison, because Scheibler's tonometer, which we use in illustration in the lectures on this subject, is based on that tonic.

173. The relation of the successive tones of the harmonic scale to any tone assumed as a fundamental is as follows; taking as the prime that whose vibrational number is 256, we have

Prime or fundamental,	256	vibrations,	or	c;
1° Harmonic,	512	"	"	c', octave;
2° "	768	"	"	g', fifth in 1st octave;
3° "	1024	"	"	c'', second octave;
4° "	1280	"	"	e'', maj. third in 2d oct.;
5° "	1536	"	"	g'', fifth of 2d octave;
6° "	1792	"	"	$a''+$, lying between 6th and 7th of 2d oct.;
7° "	2048	"	"	c''', third octave;

and so on. These harmonics are called *overtones* or *upper partials*, and, as seen above, bear a close relationship to the prime. When the prime is sounded and the upper partials exist at the same time, the resulting tone will have a determinate quality. And if the partials be successively stopped out, the quality will undergo a change, until we reach the simple tone due to the prime alone. The successive curves which represent these tones graphically will approximate gradually to that of the harmonic curve of the wave length of the prime, which it ultimately reaches when all of the partials are wanting. The wave lengths of the above curves are each equal to that of the prime.

174. It can be experimentally shown that a stretched cord, when plucked from its position of rest, will give a compound tone, which is made up of its fundamental united to some of its overtones. The educated ear can readily distinguish the existence of these simple tones, which, sounding together, determine the quality of the compound tone. But to demonstrate to the untrained ear the existence of these partial tones, it is necessary to make use of certain appliances called *resonators*, whose action depends on the

principle of *sympathetic resonance.* These consist of metal or other hollow bodies, generally spherical in form, closed except at two places; one of the openings is to permit the mass of air within to be affected by the vibration of the air without, and the other to permit the air within to be brought into near contact with that in the aperture of the ear.

175. ***Sympathetic Resonance.*** If a body capable of taking up an oscillatory motion of definite period be subjected to a series of periodic impulses, whose period is the same as that of the body considered, the aggregate effect will in time become sensible, however weak the impulses may be. But if the period of the impulses be even slightly different from that of the body, the resultant effect will, in general, never become appreciable; for, while the kinetic energy is increased by the elementary quantities of work due to the impulses applied, soon the succeeding impulses will be delivered in a direction contrary to the motion of the body, and the kinetic energy will be correspondingly diminished. The maximum energy can then never exceed a small definite quantity, and in reaching this state the body will pass through alternations of rest and motion. To determine the effect of any periodic impulse upon a body capable of being put into vibration, we have the following rule, due to Helmholtz: Resolve the periodic motion of the impulse into its component simple pendular vibrations; if the periodic time of any one of these vibrations is equal to the periodic time of the body acted upon, sensible vibration will result, and not otherwise.

176. Now consider the mass of air within the tube AB, while a simple vibratory motion, due to a simple tone, occurs in the external air. Let V be the velocity of wave propagation in the air under consideration, and n the vibrational number of the body. Then, during the first semi-vibration, the molecules at B describe half their orbits while undergoing condensation, which is transmitted through the intervening molecules to A and back to B, provided

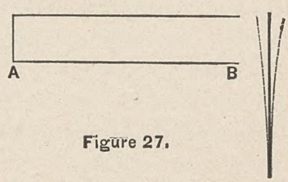

Figure 27.

$$BA + AB = \frac{V}{2n}.$$

During the second semi-vibration, the rarefaction at B will be transmitted in the same manner, and the orbits at B will be completed. Should BA be either $>$ or $< \frac{V}{4n}$, the second impulse would reach B after or before its molecular orbits had been completed. Under these circumstances, succeeding impulses would in a short time reduce the displacements of the molecules to zero, and never permit them to attain an appreciable value, and therefore the vibration of the air column would not give a sound of appreciable intensity. But if, on the contrary, the impulses were of the same periodicity as the air molecules, each successive impulse would add to the first displacement, and this addition would continue until the work of the resistances developed was exactly equal to the increment of energy caused by each impulse. The displacements of the molecules would then have attained their maximum value, and the resulting sound a fixed intensity.

177. Each confined mass of air has a particular periodicity, and each of the resonators of Helmholtz is carefully contrived to respond to a given periodicity of vibratory motion. If, then, by the rule above given, any composite sound exist, and one of these resonators be applied to the ear, the resonant effect will indicate whether the simple tone corresponding to the resonator is present or absent in the composite sound. This and analogous experiments show that sympathetic vibration is not due to any property peculiar to the ear, but that it is a mechanical effect separate and distinct from the sense of audition.

178. The energy of motion depending upon the mass and velocity, we see clearly that of two sounding bodies, vibrating with the same amplitude, the smaller mass will more quickly give up its energy to the surrounding air and sooner cease sounding. Tuning-forks being generally made of steel, will, when put into rather strong vibration, continue sounding for a reasonable length of time. When mounted upon their resonant boxes, the latter containing a mass of air capable of vibrating in unison with it, they affect larger masses of air than when not so mounted, and come more quickly to rest; but the sound will have greater intensity, and can the more readily be used to study the phenomena of sympathetic resonance. If such a tuning-fork be in the vicinity of a vibrating sounding body whose sound contains the tone of the fork, the latter will in

time indicate the fact by coming into sympathetic vibration. The analysis, then, of any composite note can be practically made by means of a sufficient number of such forks, whose vibrational numbers embrace all the simple notes of the composite sound. Conversely, the synthesis of a composite note can be effected by setting in vibration all the forks, with proper amplitudes, which the analysis indicates belong to the note in question.

179. When plates, bells, strings, etc., are put into vibration, they may either vibrate as a whole, or separate into parts which vibrate two, three, four, or more times as rapidly; or both of these conditions may occur simultaneously. Each of the simple periodic vibrations has an actual existence, and corresponds to a single musical tone of definite pitch, which may be recognized as above described.

180. In listening for any simple tone in the composite note, it is important to clearly fix the attention upon the special tone whose existence is to be determined, and for this purpose the tone should be sounded alone before listening for it in the composite note. When sufficiently practiced in this manner, the ear can readily acquire the faculty of detecting them without the use of resonators.

181. By means of the *monochord*, which consists essentially of a string stretched over two bridges on a sounding-box, we can verify the simultaneous existence of the prime and upper partials, and estimate the influence of the latter in affecting the quality of the sound. The theory of vibrating strings shows that the frequency of vibration of the same string under the same tension is inversely proportional to its length. Plucking the string at its centre, the resulting tone will be that of its prime, modified by some of the upper partials, those of the latter being absent that require the middle point as a point of rest. By a movable bridge, the string can be divided into its aliquot parts, which being set in vibration, will give the upper partials in succession. Becoming thus acquainted with these simple tones, we can verify their presence or absence in each special case. For example, if the string be plucked at one-fourth its length, theory requires the presence of the first upper partial with the prime, and the fact will be made manifest by damping the string at the middle point immediately after plucking, when the octave will sing out, no longer encompassed by the prime.

182. These and the facts of sympathetic resonance show that the analysis of all resonant motion into simple pendular vibrations is real and actual, and that any other analysis is highly improbable. The analogous property of the ear is expressed by the law of G. S. Ohm, viz., *that the human ear perceives pendular vibrations alone as simple tones, and resolves all other periodic motions of the air into a series of pendular vibrations, hearing the simple tones which correspond to these simple vibrations.* We may therefore conclude, that in all cases whenever any motion of the air caused by a sounding body contains a simple vibration of the same periodicity as that of any other body, the latter will in time take up a vibratory motion which, if of sufficient intensity, will affect the ear with a simple musical tone of a definite pitch; and the mechanical effect of vibration will ensue, whether it be of sufficient amplitude to produce a sonorous effect or not.

183. ***Velocity of Sound in any Isotropic Medium.*** The air is the medium of transfer to the ear of the vibratory motion of a sounding body. Under a given temperature and density, its elastic force is constant in all directions, and it is therefore an isotropic medium. Being compressible, the motions of its molecules, during the passage of a sound wave, are to and fro along the line of wave propagation. They are then longitudinal vibrations, and Eq. (116), for the velocity of wave propagation, for waves with such vibrations in an isotropic medium, is

$$V^2 = \frac{\lambda^2}{2\pi^2}\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\sin^2\frac{\pi}{\lambda}\Delta x. \qquad (206)$$

184. The wave lengths of sound in air can never be greater than 54.6 ft., nor less than 0.027 ft.; for the usual sounds the limits are 27.3 ft. and 0.273 ft., at 0° C. In the above equation, Δx is the distance separating two adjacent molecules, and without knowing its absolute value for any degree of pressure, we may say that λ, even in the minimum sound wave, is very great with respect to Δx. Therefore the arc is approximately zero, and may be substituted for $\sin\frac{\pi\Delta x}{\lambda}$ without appreciable error. We then have

$$\left.\begin{aligned} V^2 &= \frac{\lambda^2}{2\pi^2}\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\frac{\pi^2\,\Delta x^2}{\lambda^2} \\ &= \frac{1}{2}\,\Sigma\mu\left[\phi(r) + \psi(r)\frac{\Delta x^2}{r^2}\right]\Delta x^2. \end{aligned}\right\} \qquad (207)$$

Hence, with the supposition of small displacements, etc., the velocity of wave propagation of sound in air is theoretically independent of the wave length, and all sounds, whether grave or acute, will travel, in air of constant pressure and temperature, with equal velocities.

Omitting the term containing Δx^4 as being small compared with that of which Δx^2 is a factor, and replacing $\phi(r)$ by its equal $\frac{f(r)}{r}$, we have

$$V^2 = \frac{1}{2}\,\Sigma\mu\,\frac{f(r)}{r}\,\Delta x^2. \qquad (208)$$

185. Let E represent the modulus of longitudinal elasticity of air, P the barometric pressure, l the length of the air column without pressure, and λ the compression due to P. Then, by Eq. (1), we have

$$E = \frac{l}{\lambda}\,P. \qquad (209)$$

Since, if the pressure P be removed, the expansion would be indefinitely great, the compression λ is sensibly equal to l, and therefore

$$E = P; \qquad (210)$$

that is, the elastic force of the air is that due to the barometric pressure on the unit area.

186. In Eq. (208), $\frac{1}{2}\,\Sigma\mu\,f(r)$ is the acceleration due to the aggregate elastic forces developed in the molecules μ by the arbitrary displacement of the molecule m, and reciprocally is the elastic acceleration of m; hence we have, by multiplying by m, the intensity of the elastic force acting on m,

$$\frac{1}{2}\,m\,\Sigma\mu\,f(r);$$

$\frac{\Delta x}{r}$ is the cosine of the angle made by this force with the axis of x, which, since the medium is isotropic, is equal to unity; multiplying the elastic intensity on m by the factor $\frac{1}{\Delta x^2}$, we have the elastic intensity on the unit area, or

$$\frac{1}{2} m \, \Sigma \mu \, f(r) \, \frac{1}{\Delta x^2} = E; \tag{211}$$

whence,

$$\frac{1}{2} \Sigma \mu \, f(r) = \frac{E \, \Delta x^2}{m}. \tag{212}$$

Substituting in Eq. (208), we have

$$V^2 = \frac{E \, \Delta x^3}{m}; \tag{213}$$

Δx^3 is the volume of the molecule, and replacing it by its equal $\frac{m}{D}$, and extracting the square root, we will have, finally, for the velocity of wave propagation in air or any gas, subjected to the law of Mariotte,

$$V = \sqrt{\frac{E}{D}}, \tag{214}$$

or directly proportional to the square root of the ratio of the elasticity of the medium to its density.

187. This conclusion is deduced on the hypothesis of the direct ratio of the elastic force to the density, and if the law of Mariotte were true for all circumstances of pressure, temperature, and density, this theoretical velocity and the actual velocity determined by experiment would perfectly accord. But this relation is true only for a perfect gas and for constant temperature.

188. The relation of these two parameters of air, considered as a perfect gas, are given by the following formulæ:

$$p'd = pd', \tag{215}$$

$$p'd = pd' \, (1 + \alpha\theta), \tag{216}$$

$$p' = p \left(\frac{d'}{d}\right)^{\gamma}, \tag{217}$$

in which p and p' are respectively the old and the new pressures or

the corresponding elastic forces, d and d' the old and the new densities; α the coefficient of expansion, a constant, and equal to $\frac{1}{273}$ for Centigrade scale; and θ, degrees of temperature Centigrade.

The first of these equations is the mathematical expression of Mariotte's law; the second, of that of Charles or Gay-Lussac; and the third, of that of Poisson. The gas to which these equations are applicable is supposed to be a perfect fluid, devoid of friction, and to have the pressure at each point uniform in all directions.

The temperature is supposed constant during all changes of pressure and density in Mariotte's formula, while in that of Charles the gas takes the pressure and density determined by the change of temperature. The formula of Poisson supposes the gas subjected to sudden changes of density, and that the heat developed, whether considered positively or negatively, is not conveyed by radiation or conduction to other bodies, or, in other words, that the quantity of heat in the gas is constant. Remembering that sudden condensation in air or gas produces heat, and sudden rarefaction cold, and assuming that these alternations are so rapid that neither the heat nor the cold is conveyed to the other particles, within the volume considered, much beyond the point at which they originate, we see that this heat and cold will produce an elastic force of greater intensity than that in either of the other two cases; therefore the value of the velocity of propagation will be greater than that given in Eq. (214), which was deduced under the supposition of the simple ratio of the elastic force to the density expressed by $\frac{p}{d}$. It might be supposed that the influence of the heat produced in the condensation of the sound wave would be neutralized by the cold produced in the rarefaction, and that therefore the resultant effect would be zero. This, however, is not the case; for the heat in the condensation has increased the difference of elastic force between the condensed stratum and the one in its front, and hence has increased the velocity, while the cold in the rarefaction has caused an equal difference between the rarefied stratum and the one in rear, and has thus added an equal increment of velocity to this portion of the wave. This is true for each stratum affected by the sound wave. Hence the disturbance passes each stratum of the condensed and rarefied portions with the same velocity, and this may be regarded as the velocity of the wave.

189. Since the vibrational number of sound waves varies between 20 and 40000, for the extreme limits, the alternate condensations and rarefactions occur with sufficient rapidity to necessitate the application of the formula of Poisson for the determination of the velocity of sound in air and in other gaseous media.

190. ***Pressure of a Standard Atmosphere.*** Let p be the pressure of the atmosphere when the barometric column corresponds to 76 cm., the mercurial density being 13.5962, and g 981 dynes; we then have

$$p = 981 \times 13.5962 \times 76 = 1.01368 \times 10^6 \text{ dynes}, \tag{218}$$

as the corresponding pressure of the atmosphere upon a square centimetre. But since the density of mercury, referred to the standard at the same locality, is independent of the locality, and hence independent of g, we may assume as the *standard atmosphere* that whose pressure on the square centimetre at all localities is equal to 10^6 dynes. Hence,

$$p = g \times d_m \times h = 10^6 \text{ dynes}. \tag{219}$$

By substituting in this equation the value of g for the latitude of the place, and solving with reference to h, we will determine the barometric height corresponding to the standard atmosphere at that locality; g varies from 978.1 dynes at the equator to 983.11 dynes at the pole.

191. ***Height of the Homogeneous Atmosphere.*** If the atmosphere be supposed replaced by an atmosphere of uniform density D, as that of standard dry air at 0° C., and height H, exerting the same pressure, H may be obtained from the equation

$$p = g \cdot D \cdot H = 10^6 \text{ dynes}; \tag{220}$$

from which we have

$$\left. \begin{aligned} H &= \frac{10^6}{g \cdot D} = \frac{10^6}{981 \times .0012759} = 7.9894 \text{ cm.} \times 10^5 \\ &= 7989.40 \text{ m.} = 26212.18 \text{ ft.}, \end{aligned} \right\} \tag{221}$$

which is constant at the same locality, for the same temperature

and barometric height. If the temperature become $\theta°$ C., we have, by the law of Charles or Gay-Lussac,

$$H' = (1 + \alpha\theta)\,H = H\alpha\tau, \tag{222}$$

in which τ is the absolute temperature, and α the coefficient of expansion.

192. Replacing the elastic force E by its equal, in terms of the homogeneous atmosphere, in Eq. (214), we have

$$V = \sqrt{\frac{E}{D}} = \sqrt{\frac{H\alpha\tau \cdot g \cdot D}{D}} = \sqrt{H\alpha\tau g}, \tag{223}$$

which is Newton's formula for the velocity of sound in air. Making $\tau = 273°$, corresponding to zero Centigrade, and $g = 981$ dynes, we have

$$V = \sqrt{7.9894 \times 10^5 \times 981} = 2.8 \times 10^4 = 280.0 \text{ metres}. \tag{224}$$

For any other temperature, we have

$$V = \sqrt{7.9894 \times 10^5 \times 981 \times \alpha\tau} = 280\sqrt{1 + \alpha\theta}. \tag{225}$$

193. These values of the velocity of sound in air are about one-sixth less than those determined by experiment, the discrepancy being due to the supposition that Mariotte's law expresses the relation of pressure and density. The law of Poisson is, however, applicable; hence we have

$$p' = p\left(\frac{D'}{D}\right)^\gamma;$$

differentiating, $$dp' = \gamma p\left(\frac{D'}{D}\right)^{\gamma-1}\frac{dD'}{D},$$

$$\frac{dp'}{dD'} = \gamma\frac{p}{D}\left(\frac{D'}{D}\right)^{\gamma-1} = \gamma\frac{p'}{D'}. \tag{226}$$

Whence we see that when a sound wave is passing through air, the ratio of the increment of the elastic force to that of the density is equal to the ratio of the elastic force to the density, multiplied by the constant γ. The value of γ can be determined from a direct observation, by accurately measuring V, α, and θ, and substituting in the equation

$$V = \sqrt{\gamma \frac{p'}{D'}} = 10^4 \times 2.80 \sqrt{\gamma (1 + \alpha\theta)}, \tag{227}$$

and solving with respect to γ. Its value has been found to be, approximately, 1.41 for all simple gases not near their points of liquefaction. The final formula, therefore, is

$$\left. \begin{aligned} V &= 332.64 \text{ m.} \times \sqrt{1 + \alpha\theta} \\ &= 1091.35 \text{ ft.} \times \sqrt{1 + .003660\theta^\circ}, \end{aligned} \right\} \tag{228}$$

for the velocity of sound in air at the locality where $g = 981$ dynes, barometric height 76 cm., and temperature θ° Centigrade.

194. At West Point, assuming the barometric height to be 76 cm., and $g = 980.3$ dynes, we have, for the velocity of sound in air at any temperature,

$$\left. \begin{aligned} V &= \sqrt{980.3 \times 7.9894 \times 1.41 \times 10^5 \times \alpha\tau} \\ &= 332.3 \text{ m.} \times \sqrt{1 + \alpha\theta} \\ &= 1090.23 \text{ ft.} \times \sqrt{1 + \alpha\theta}. \end{aligned} \right\} \tag{229}$$

Since the value of $\alpha = \frac{1}{273}$, we see that the velocity increases nearly 2 feet for each degree Centigrade, and hence is greater in warm than in cold weather, all other things being equal. At 60° F., we may take the velocity of sound in air to be approximately 1123 feet per second.

195. The value of the velocity of sound in any gas can, in like manner, be obtained theoretically by substituting in the equation

$$V = \sqrt{\gamma \frac{p'}{D'}}, \tag{230}$$

for D' the density of the gas referred to that of air as unity, and for p' the value of the pressure in terms of the barometric height, γ being taken as 1.41; or it may be obtained more simply by dividing

$$V = 332.3 \text{ m.} \times \sqrt{1 + \alpha\theta} \tag{231}$$

by the square root of the density of the gas referred to air as unity.

At zero degrees Centigrade, we have for the theoretical value of the velocity of sound in the following gases:

Air,	332	Carbon dioxide,	262
Hydrogen,	1269	Carbon monoxide,	337
Oxygen,	317	Olefiant gas,	314

196. ***Velocity of Sound in Air and other Gases, as affected by their not being Perfect Gases.*** The formulæ of Mariotte, Charles, and Poisson are only applicable to perfect gases. This condition requires the elasticities to be perfect, and the excess of the elastic force which gives rise to wave propagation to be indefinitely small when compared with the elasticity of the gas in its quiescent state.

A series of experiments made by Regnault, the results of which are given in the Comptes Rendus, Vol. 66, page 209, show that these conditions are not fulfilled, and that the theoretical velocity therefore differs from the actual. The sounds were made in tubes of different cross-section, by discharging a pistol with different charges of powder. The results are grouped in the following table:

Diameter of Tube, 0.108 m. Length, 566.74 m.				Diameter of Tube, 0.3 m. Length, 1905 m.					Diam. of Tube, 1.10 m.	
Charge, 0.3 gr.		Charge, 0.4 gr.		Charge, 0.3 gr.		Ch'ge, 0.4 gr.	Charge, 1.5 gr.		Charge, 1.00 gr.	
Distances.	Mean Velocities.	Distances.	Mean Velocities.	Distances.	Mean Velocities.	Mean Velocities.	Distances.	Mean Velocities.	Distances.	Mean Velocities.
566.74	330.99	1351.95	329.95	1905	331.91	332.37	3810.3	332.18	749.1	334.16
1133.48	328.77	2703.00	328.20	3810	328.72	330.34	7620.6	330.43	920.1	333.20
1700.22	328.21	4055.85	326.77				11430.0	329.64	1417.9	332.50
2266.96	327.04	5407.80	*323.34				15240.0	328.96	2835.8	331.72
2833.70	327.52								5671.8	331.24
									8507.7	330.87
									11343.6	330.68
			* Uncertain.						14179.5	330.56
									17015.4	330.50
									19851.3	330.52

197. From these results we see: 1°, that the mean velocity of the same wave decreases from the origin; 2°, that it is less for the same charge and route in tubes of smaller diameter; 3°, that it

decrease less rapidly in tubes of larger diameter. Regnault also, by means of sensitive diaphragms, followed the course of the waves after they became inaudible, and obtained similar results with respect to these. He found that a sound produced by a pistol discharge, of one gramme of powder, became inaudible at distances of 1150, 3810, and 9540 metres, in tubes of 0.108 m., 0.30 m., and 1.10 m. diameter, respectively, and that the waves became insensible after traveling distances of 4056, 11430, and 19851 metres respectively. In the tube of 1.1 m. diameter, with a charge of 2.4 grains, the wave ceased to be audible at 58641 metres, and ultimately ceased at 97735 metres. These distances of audibility are, approximately, directly proportional to the diameters of the tube.

198. The mathematical theory discusses the case of a perfect gas, and assumes that the propagation in an indefinite tube is continuous. The above experiments show that this is not really the case. The assumptions made by implication in a perfect gas are:

1°. That the laws of Mariotte, Charles, and Poisson are true, but it is well known that no gas obeys exactly these laws.

2°. That its elasticity is unaffected by admixture with other gases.

3°. That the gas offers no opposition by its inertia to wave transmission; but experiment shows that an intense disturbance always produces a real motion of the surrounding particles, which increases the velocity, especially within sensible distances from the origin. Such is the case, no doubt, in cannon discharges, violent lightning-flashes, and other like instances.

4°. Theory supposes the excess of pressure due to a vibrating body small, in comparison with the quiescent barometric pressure; but in the cases cited above, the excess of pressure at the origin may be large, and hence cause an increase in the value of V near the origin. Therefore, the correction of Art. 193, called that of La Place, in such cases is not exact.

199. Regnault ascribes as the principal cause of the diminution of the intensity, the loss of kinetic energy by the reaction of the sides and ends of the tube, and confirms this by the fact that the sounds are quite audible outside the tube during their first passage, and in a less degree at each succeeding passage. As a secondary cause, he ascribes the influence of the walls of the tube in diminishing the elasticity without affecting the density. This is con-

firmed by the fact that in the above experiments, where the waves have been produced by the same charge, and hence have the same sensibility at the origin, they have not the same intensity after traveling over equal routes. The *mean* limiting velocity ought, therefore, to be the same, if the weakening is due to the loss of mv^2 on account of the sides. The experiments show that this is not the case; hence, the sides exercise another effect on air different from that indicated as the principal cause of the diminution of the intensity, an action affecting the elasticity and not the density. In free air this effect would be null, and in the tube of 1.1 m. it is taken as approximately so. The mean velocity of propagation, *in dry air at* 0° C., of a wave produced by the discharge of a pistol, and estimated from the origin to the point at which its sensibility can no longer be appreciated by the ear is, according to Regnault's experiments,

$$V = 330.6 \text{ m.}$$

The mean limiting velocity, considered from the origin to the point at which its existence can no longer be detected upon a sensitive diaphragm, is

$$V = 330.3 \text{ m.},$$

which differs from the mean limiting velocity in the 1.1 m. tube by only 0.32 m.

200. *Velocity of Sound in Gases independent of the Barometric Pressure.* Since an increase in the barometric pressure increases the elasticity and density in the same proportion, theory indicates that no change, due to this cause alone, will take place in the velocity. The experiments of Stampfer and Myrbach in the Tyrol, in 1822, between two stations whose difference in altitude was 1364 m., and of Bravais and Martins in Switzerland, in 1844, between two stations whose difference of level was 2079 m., indicated no variation in the velocity, due to the change in the barometric pressure. Regnault's experiments upon air in the tube 0.108 m. in diameter, over a distance of 567.4 m., with pressures varying from 0.557 m. to 0.838 m., and over a distance of 70.5 m., with pressures varying from 0.247 m. to 1.267 m., found no variation in the velocity, due to this cause.

The theoretical ratio of the velocities of sound in gases, given by

$$\frac{V'}{V} = \sqrt{\frac{D}{D'}}, \qquad (232)$$

was experimentally confirmed to a near degree of approximation in the cases of hydrogen, carbon dioxide, and air. The tube 0.108 m., filled for a length of 567.4 m., gave for hydrogen 3.801 m., for carbon dioxide 0.7848 m., which differ but little from the theoretical values 3.682 m. and 0.8087 m., the velocity in air being taken as unity. Hence the formula may be taken as an expression for the limiting law. The determination of the velocity of sound in free air was made by means of reciprocal cannon discharges. There were two series of these experiments. For the first, consisting of 18 discharges, the membrane being 1280 metres distant, the mean velocity, referred to dry air at 0° C., was found to be

$$V = 331.37 \text{ m.}$$

For the second series, of 149 discharges, over a distance of 2445 m., during 11 days of trial, with the temperature of the air varying from 1.5° to 21.8° C., and with great variations in the wind, the mean velocity, referred to dry air at 0° C., was

$$V = 330.7 \text{ m.},$$

a sensible diminution of the velocity, due to the increased distance.

201. ***Velocity of Sound in Liquids.*** The value of the velocity of sound in liquids is likewise given by the general formula

$$\left.\begin{aligned} V &= \sqrt{\frac{gd_mH}{D\lambda}} = \sqrt{\frac{gd_mH}{\lambda} \times \frac{1}{D}} \\ &= \sqrt{E \times \frac{1}{D}} = \sqrt{\frac{E}{D}}, \end{aligned}\right\} \qquad (233)$$

in which H is the arbitrary barometric height, d_m the density of mercury, and g the acceleration due to gravity. The numerator is then the pressure due to the height of the barometer, and when divided by λ, which is the diminution of the volume due to the increase of pressure, gd_mH gives the ratio of the pressure to the corresponding compression, and is therefore the measure of the elastic force of the medium. The square root of this quantity, divided

by the square root of the density, will be the value of the velocity of sound in the liquid.

202. Colladon and Sturm made a series of experiments to determine the actual value of the velocity of sound in water, in Lake Geneva, in the year 1826. The sound was caused by the strokes of a hammer upon a bell submerged one metre below the surface, and so arranged that the epoch of the stroke could be determined by a flash of powder. The instant of hearing the sound was indicated by a stop-watch to within one-quarter of a second. The distance traveled by the sound was found to be 13487 m. to within 20 m., and the time of this travel, from a mean of many experiments, was found to be 9.4 s. The temperature of the water was 8.1° C., its density at that temperature, referred to that of water at the standard temperature, was unity plus a negligible fraction, its compressibility was taken at .0000495, and the barometric height at 76 cm. The density of mercury referred to the same temperature is 13.544, and $g = 9.8088$.

Making these substitutions in the preceding formula, we find

$$V = \sqrt{\frac{9.8088 \times 13.544 \times .76}{.0000495}} = 1428 \text{ m.}$$

The actual velocity found was $\frac{13487 \text{ m.}}{9.4} = 1435$ m., differing from the theoretical value but 7 m. The latter may itself vary within wider limits, on account of the inexactness of the value of the compressibility of water, whose most probably correct value, from the experiments of Regnault, is assumed to be .00004685.

203. The principal facts derived from these experiments of Colladon are (Tome XXXVI, Annales de Chimie) that at distances beyond 200 metres the quality of the sound is changed, and the sensation is similar to the quick, brief noise produced by the striking together of two knife-blades in air. The diminution of intensity with the distance is noticed, and at short distances, greater than 200 metres, it is not possible to tell whether the sound originates at a near origin of weak intensity, or at a distant origin with increased intensity. The duration is less than in air; as it should be from its value $\frac{\lambda}{V}$, λ being greater and V being smaller in air

than in water. When the vibrations proceeding from the sounding body reach the surface of the water at great angles of incidence, the sound does not pass into the air. At distances greater than 400 to 500 metres, the ear in air does not hear the sound originating in the water. At 200 metres the sound is readily heard. In these experiments, the bell being placed 2 metres below the surface, the angle of incidence at 400 metres is approximately 89° 43′; at 200 metres, 89° 26′.

Finally, the existence of a sharper acoustic shadow shows that the wave lengths are proportionally shortened in water compared with the waves made in air by the same sounding body.

204. ***Velocity of Sound in Solids.*** The ordinary solids upon which experiments have been made for the determination of the velocity of sound are glass, the various metals, and wood. In the latter, from the manner of its growth in the tree, the three directions, along the axis, in the direction of the radius, and normal to the plane of these two, possess necessarily different elasticities. The coefficients of elasticity also differ in different species, and in the same species, when grown in different localities, under different circumstances of soil, temperature, and moisture. Reasonably exact determinations belong then only to the particular specimen experimented upon, and mean values are usually taken for any one kind of wood in a given direction. In metals and glass, variations of the coefficients arise from the methods of their manufacture, and modifications result from every circumstance which affects their density and other physical properties. None of the solids can be said to be perfectly homogeneous; but on the assumption that they are approximately so, different experimenters have obtained values for their coefficients which do not vary between very wide limits.

205. In solids, the sound may result either from transversal or from longitudinal vibrations. In the cases here considered, the vibrations are understood to be longitudinal, that is, the molecular displacements are in the direction of the propagation.

When a solid bar, taken as homogeneous, transmits a longitudinal vibration, the velocity of the propagation has been found to be given by the equation

$$V = \sqrt{\frac{g}{\lambda}}, \qquad (234)$$

in which λ is the elongation due to the weight of the bar. Substituting for λ its value in terms of Young's modulus,

$$\lambda = \frac{1}{E} \cdot \frac{Pl}{s}, \tag{235}$$

and making s equal to one square centimetre, l equal to one metre, and P the weight of the bar, we have

$$V = \sqrt{\frac{E}{D}}, \tag{236}$$

the same in form as has been found for gases and liquids.

206. Different methods have been employed to find E, viz., by the direct method of elongations or compressions, by flexure, by transversal and by torsional vibrations of the bar. The values given for the different metals, in Art. 23, have been obtained by Wertheim, by the method of elongations. Could we accurately determine the velocity of sound in solids by direct experiment, the value of E could be readily found by the solution of the above equation. But this velocity being very great compared with that in air, and because of the impracticability of finding sufficiently long homogeneous lengths, an accurate determination of E by this means is impossible. Biot, by a direct experiment on 951 metres in length of cast-iron pipe, found that the velocity was 10.5 times that in air; but the want of homogeneity, due to the numerous leaded joints, without doubt influenced this result appreciably. Wertheim found about the same value in wrought iron, by experimenting upon 4067.2 metres of telegraph wire.

207. Assuming the experimental values for E given in Art. 23, and taking g to be 981 dynes, the velocities of sound are, by the above formula, found to be as follows:

	E.	D.	V IN CENTIMETRES.	RATIO TO V IN AIR.
Lead, . .	$177 \times 981 \times 10^6$	11.4	1.23×10^5	3.7
Gold, . .	$813 \times 981 \times 10^6$	19.0	1.74×10^5	5.3
Silver, . .	$736 \times 981 \times 10^6$	10.5	2.61×10^5	8.0
Copper, . .	$1245 \times 981 \times 10^6$	8.6	3.56×10^5	10.7
Iron, . .	$1861 \times 981 \times 10^6$	7.0	5.13×10^5	15.5
Steel, . .	$1955 \times 981 \times 10^6$	7.8	4.99×10^5	15.0

For glass, with density of 2.94, V has been found to be, by the same method, 4.53 cm. $\times 10^5$; and for brass, of density of 8.47, $V = 3.56$ cm. $\times 10^5$; or 13.6 and 10.8 times the velocity in air, respectively.

208. The following velocities of sound in wood, deduced from the observations of Wertheim and Chevandier (Comptes Rendus, 1846), are taken from "Everett's Physical Constants," page 65, from which also several of the above numbers have been obtained:

	Along Fibre.	Radial.	Tangential.
Pine, ...	3.32×10^5	2.83×10^5	1.59×10^5
Beech, ...	3.34×10^5	3.67×10^5	2.83×10^5
Birch, ...	4.42×10^5	2.14×10^5	3.03×10^5
Fir, ...	4.64×10^5	2.67×10^5	1.57×10^5

209. The preceding values of the velocities of sound in solids are true only when the medium is in the form of a bar of small cross-section. Wertheim has shown by his investigations, based on the theory of Cauchy, that the corresponding velocities in extended homogeneous solids are greater than the above results in the ratio of

$$\sqrt{\frac{3}{2}} \ : \ 1.$$

210. ***Reflection and Refraction of Sound.*** The laws deduced in Art. 77 for the reflection and refraction of wave motion are applicable to the undulations of sound. From the equation

$$\sin \phi = \mu \sin \phi', \tag{237}$$

the direction of any deviated ray or that of any deviated plane wave by a plane surface, can be found when $\frac{V}{V'}$ is substituted for μ. If $V' > V$, then $\phi' > \phi$, and the refracted ray is thrown from the normal; conversely, if $V' < V$, then $\phi' < \phi$, and the refracted ray is bent towards the normal. A ray of sound in air, incident on the surface of water, will be refracted, provided the angle of incidence be less than 13° 26′; for since V' in water is about 1428 m., and V in air about 332, we have

$$\mu = \frac{332}{1428} = .2325,$$

and $\sin \phi = .2325,$ or $\phi = 13° 26'.$

For greater incidences the ray is totally reflected, and does not enter the water.

211. *Consequences of the Laws of Reflection.*

1°. If a sound originate at one of the foci of an ellipsoid, it will be reflected to the other focus.

2°. If at the focus of a paraboloid, the rays of sound will be reflected in lines parallel to the axis, and can be again collected at the focus of another similar paraboloid, with sensibly undiminished intensity. The slightest sound, as the ticking of a watch, may be employed to illustrate this case of reflection.

3°. The speaking-trumpet and speaking-tube are employed to prevent the too rapid dissipation of sound. The former, partly by reflection from its sides and largely by resonance, concentrates the sound within the volume of the cone whose apex is the mouth-piece and whose section is that of the other end of the trumpet. The speaking-tube confines the energy in the narrow compass of the tube, the loss being insignificant in the ordinary lengths employed.

4°. When a sound is reflected by any obstacle which prevents its direct transmission, and the observer is at such a distance that the direct and reflected sounds are not confounded, the reflected sound is called an *echo*. Thus, if A be the position of the observer, S the origin from which a sound of short duration emanates, and W the obstacle, such as a wall, then the direct sound will reach the observer in the time $\frac{SA}{332.4}$, and the reflected sound in the time $\frac{SW + WA}{332.4}$, the temperature being 0° C. If $\frac{SW + WA - SA}{332.4}$ be sufficiently great, so that the reflected sound arrives after the cessation of the direct sound, then the echo will be heard, provided the intensity be of sufficient value. If the two sounds commingle,

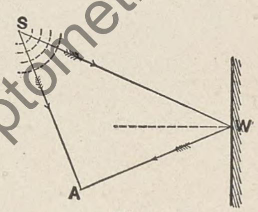

Figure 28.

the resultant sound will be prolonged, and partial resonance will ensue. The number of distinct impressions distinguished by the ear will determine the shortest difference of route necessary to establish the echo. Thus, if we take nine per second, $\frac{332}{9}$ or 37 m. is the shortest difference of route at 0° C.

5°. The conditions of interference of sound are the same as those discussed in Arts. 65–68. Hence, it is theoretically possible that two sounds affecting the ear simultaneously will result in silence, and practically it will be shown, in the lectures on this part of the course, that such an experiment is also possible. Other illustrations of interference are also reserved for the lectures.

212. ***Refraction of Sound.*** In order that the rays of sound shall converge after deviation by refraction, we see from the formula that $\mu = \frac{V}{V'}$ must be greater than unity. Then the deviated wave will, in general, become converging, and the energy accumulate on an ever decreasing surface. Examining the table, Art. 195, we see that V' in carbon dioxide is 262 m., and hence, when the incident medium is air,

$$\mu = \frac{332}{262} = 1.25,$$

and

$$\sin \phi = 1.25 \sin \phi'. \tag{238}$$

The sound lens devised by Sondhaus is a double convex lens of collodion filled with carbon dioxide, which collects the sound rays proceeding from any sonorous body and concentrates them appreciably at another point on the opposite side of the lens. By means of a concave lens of the same material, filled with hydrogen, $V' = 1269$ m., it will be evident, after the study of the properties of lenses, as explained in optics, that a similar result would be effected. The slight noise produced by the ticking of a watch may be collected by this means at a point so that the noise is audible, when without this assistance it would be inappreciable at the same point.

213. ***General Equations for the Vibratory Motion of a Stretched String.*** The bodies usually employed to produce musical sounds by their vibrations are strings, rods air-

columns, plates, bells, etc. When the vibrations of the particles are perpendicular to the direction of wave propagation, they are called *transversal*, and when in the same direction, *longitudinal*.

We will first consider the vibrations of a perfectly elastic and flexible string, supposed to be stretched between two points whose distance apart is l, by a force which produces a tension T. Let the elongation be that given by

$$l' - l = \frac{T}{E} l, \tag{239}$$

in which l is the natural length, l' the length after the tension T is applied, and E is the longitudinal modulus. If the displacements of the string from its position of rest be due to the incessant action of forces whose rectangular accelerations are X, Y, Z, these with the tension T will be the only extraneous forces considered.

Let m be the mass of any element; x, y, z, $x + dx$, $y + dy$, $z + dz$, the co-ordinates of its extremities and its length ds; α the area of its cross-section, and ρ its density; then

$$m = \rho\alpha\, ds.$$

Let the components of T at x, y, z, be

$$T\frac{dx}{ds}, \qquad T\frac{dy}{ds}, \qquad T\frac{dz}{ds};$$

and at $x + dx$, $y + dy$, $z + dz$, be

$$T\frac{dx}{ds} + dT\frac{dx}{ds}, \qquad T\frac{dy}{ds} + dT\frac{dy}{ds}, \qquad T\frac{dz}{ds} + dT\frac{dz}{ds}.$$

The general equations of motion will then be

$$\left.\begin{aligned} \rho\alpha\, ds\left(X - \frac{d^2x}{dt^2}\right) + dT\frac{dx}{ds} &= 0, \\ \rho\alpha\, ds\left(Y - \frac{d^2y}{dt^2}\right) + dT\frac{dy}{ds} &= 0, \\ \rho\alpha\, ds\left(Z - \frac{d^2z}{dt^2}\right) + dT\frac{dz}{ds} &= 0. \end{aligned}\right\} \tag{240}$$

214. These equations are simplified when we suppose that the string is arbitrarily displaced from its position of equilibrium, and

abandoned to itself, without the action of the forces X, Y, Z. It will then oscillate about its position of rest, and the only extraneous force that acts will be the tension T, whose intensity will vary between known limits. Let the axis of x coincide with the string in its position of rest, and the co-ordinates of the element m, at the time t, be $x + \xi$, η, ζ. If the displacement be supposed small, ξ, η, and ζ are functions of x and t, and x is independent of t, and the above equations reduce to

$$\left.\begin{aligned} \rho\alpha\, dx\,\frac{d^2\xi}{dt^2} - dT\,\frac{d(x+\xi)}{ds} &= 0,\\ \rho\alpha\, dx\,\frac{d^2\eta}{dt^2} - dT\,\frac{d\eta}{ds} &= 0,\\ \rho\alpha\, dx\,\frac{d^2\zeta}{dt^2} - dT\,\frac{d\zeta}{ds} &= 0. \end{aligned}\right\} \qquad (241)$$

Let T' be the tension when the string is straight, and T when the string is displaced; the length of the element is in the first case dx, and in the second ds; these are connected by the equations

$$ds = dx\left(1 + \frac{T - T'}{E}\right), \qquad (242)$$

$$ds^2 = (dx + d\xi)^2 + d\eta^2 + d\zeta^2; \qquad (243)$$

from which, when $d\eta$ and $d\zeta$ are very small, we have

$$ds = dx + d\xi, \qquad (244)$$

$$T = T' + E\,\frac{d\xi}{dx}. \qquad (245)$$

Substituting in Eqs. (241), we have

$$\left.\begin{aligned} \rho\alpha\,\frac{d^2\xi}{dt^2} &= E\,\frac{d^2\xi}{dx^2},\\ \rho\alpha\,\frac{d^2\eta}{dt^2} &= T'\,\frac{d^2\eta}{dx^2},\\ \rho\alpha\,\frac{d^2\zeta}{dt^2} &= T'\,\frac{d^2\zeta}{dx^2}, \end{aligned}\right\} \qquad (246)$$

Replacing $\dfrac{E}{\rho\alpha}$ and $\dfrac{T'}{\rho\alpha}$ by u^2 and v^2 respectively, we have

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= u^2\frac{d^2\xi}{dx^2},\\ \frac{d^2\eta}{dt^2} &= v^2\frac{d^2\eta}{dx^2},\\ \frac{d^2\zeta}{dt^2} &= v^2\frac{d^2\zeta}{dx^2}.\end{aligned}\right\} \tag{247}$$

The integration of these three partial differential equations give (Analytical Mechanics, Appendix IV),

$$\left.\begin{aligned}\xi &= F(x+ut)+f(x-ut),\\ \eta &= F(x+vt)+f(x-vt),\\ \zeta &= F(x+vt)+f(x-vt).\end{aligned}\right\} \tag{248}$$

215. The first equation determines the longitudinal vibrations, or those along the axis of the string, and the other two give the transversal vibrations along y and z respectively. Because of the independence of the differential equations, the three vibrations in general coexist and are wholly independent of each other, and since the differential equations are of the same form, we see that the two kinds of vibrations are subjected to the same laws. They may each be discussed separately. Each is due to a progressive motion forward and backward along the string. These motions may be of the most varied character, but the particular form of the motion depends on the form of the functions whose symbols are F and f. The only conditions imposed so far are that for $x = 0$ and $x = l$, ξ, η, and ζ are zero for all values of t. These, together with any assumed initial conditions, will enable us to determine the form of the functions F and f, and thus complete the solution of the problem.

216. Since the vibrations parallel to y and z are exactly alike in every particular, the discussion of one will do for the other, and we will consider that of y, given by the equation

$$\eta = F(x+vt)+f(x-vt). \tag{249}$$

Assume the conditions that at the epoch, or when $t = 0$,

$$\eta = \phi(x) \qquad \text{and} \qquad \frac{d\eta}{dt} = v\psi'(x), \tag{250}$$

in which the functions ϕ and ψ are supposed known, and that ψ' is the derived function of ψ. If $t = 0$, we have

$$\eta = \phi(x) = F(x) + f(x), \tag{251}$$

$$\frac{1}{v} \cdot \frac{d\eta}{dt} = \psi'(x) = F'(x) - f'(x); \tag{252}$$

$$\therefore \quad \psi(x) = F(x) - f(x); \tag{253}$$

and hence,

$$F(x) = \frac{\phi(x) + \psi(x)}{2}, \tag{254}$$

$$f(x) = \frac{\phi(x) - \psi(x)}{2}. \tag{255}$$

Therefore $F(x)$ and $f(x)$ are known for all values of x from 0 to l, when, as is supposed, $\phi(x)$ and $\psi(x)$ are known between the same limits.

For the extremities, we have, by placing $x = 0$ and $x = l$,

$$F(vt) + f(-vt) = 0, \tag{256}$$

$$F(l + vt) + f(l - vt) = 0; \tag{257}$$

whence, $F(vt)$ and $f(-vt)$ are equal, with contrary signs, and thus become known for all values from $t = 0$ to $t = \infty$.

217. The value of η can be expressed by means of a single function by substituting $vt + l - x$ for vt in Eq. (257); whence,

$$-F(2l - x + vt) = f(x - vt); \tag{258}$$

which in Eq. (249) gives

$$\eta = F(x + vt) - F(2l - x + vt). \tag{259}$$

Again, for vt, in Eq. (257), substitute $l + vt$; then

$$F(2l + vt) = -f(-vt) = F(vt); \tag{260}$$

whence we conclude that the function F takes the same value when the variable vt is increased by $2l$; and therefore by $4l$, $6l$, $8l$, ...

or $2nl$, n being a positive whole number. Therefore, if $F(vt)$ is known from $vt = 0$ to $vt = 2l$, its value is known for all values from $t = 0$ to $t = \infty$.

Replace vt by $l - vt$, in Eq. (257), vt being less than l; then

$$F(2l - vt) = -f(vt); \tag{261}$$

but $f(vt)$ is known for all values of vt between 0 and l; therefore $F(vt)$ is known for all values of vt between l and $2l$.

Hence, the value of $F(x + vt)$ is known for all values of $x + vt$ from 0 to ∞; and, similarly, the value of $f(x - vt)$ can be found for all values between 0 and $-\infty$; and therefore the problem is completely solved.

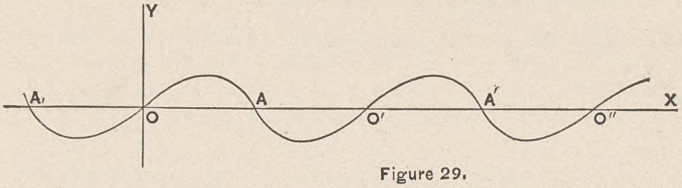

Figure 29.

218. The function whose symbol is F is subject to the following conditions, derived from Eqs. (256), (257), (260), (261),

$$F(x) = -F(-x), \tag{262}$$

$$F(l + x) = -F(l - x), \tag{263}$$

$$\left.\begin{aligned} F(x) = F(2l + x) &= F(4l + x) = \dots\dots\dots \\ &= F(2nl + x), \end{aligned}\right\} \tag{264}$$

$$F(x) = -F(2l - x) = -F(4l - x) = \dots\dots \tag{265}$$

From Eq. (262) we see that the curve represented by $\eta = F(x)$ is continued in similar forms on each side of O in the figure; from Eq. (263), that the forms are similar on each side of A; from Eq. (264), that the form is repeated from O′ to O″ exactly as from O to O′; and from Eq. (265), that the form of the curve inverted is the same from O′ to A as from O to A.

The motion of any particle is that of oscillation about its place of rest, and of which the period is $\frac{2l}{v}$. This vibratory motion is gradually diminished, while the period remains unchanged, because

of the energy communicated to the air, and through the points of attachment to other bodies. The time of one complete oscillation is

$$t = \frac{2l}{v} = 2l\sqrt{\frac{\rho a}{T'}}, \tag{266}$$

and the number of oscillations in the unit of time is

$$n = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T'}{\rho a}}. \tag{267}$$

Therefore in the transversal vibrations of a string, the resulting pitch is inversely proportional to its *length*, directly as the square root of the *tension* when straight, and inversely as the square root of the *density* by the *area of cross-section*.

219. The number of longitudinal oscillations in the unit of time is

$$n' = \frac{u}{2l} = \frac{1}{2l}\sqrt{\frac{E}{\rho a}}, \tag{268}$$

whence the pitch depends only upon the *length* of the string and the *material* of which it is made, and is independent of the tension, unless the latter should be so considerable as to change the value of E. Experiment appears to indicate that the longitudinal pitch increases slightly with the tension; but this may be accounted for in the elongation experienced, which is always accompanied with a slight diminution of density ρ, and should this occur, the formula indicates that the pitch should rise.

The ratio of the numbers for the same string is given by

$$\frac{n'}{n} = \sqrt{\frac{E}{T'}} = \sqrt{\frac{l}{\Delta l}}. \tag{269}$$

M. Cagniard Latour experimented on a cord of 14.8 m. in length, and found

$$\frac{n'}{n} = \frac{188}{7} \quad \text{and} \quad \Delta l = 0.05 \text{ m.}$$

Substituting in the formula, we have

$$\frac{188}{7} = \sqrt{\frac{14.8}{\Delta l}};$$

whence, $\Delta l = 0.052$ m., a sufficiently near approximation.

220. The preceding values of n and n' are the least numbers of transversal and longitudinal vibrations of the string, and therefore correspond to its fundamental tones; but we know that each of the vibrations is decomposed into any number of vibrations of equal periodicity, when the string is divided into a like number of symmetrical parts. This can be shown more readily when the integral equation is expressed in a series which is a function of sines and cosines. Thus, it is evident that a possible solution of the differential equation

$$\frac{d^2\eta}{dt^2} = v^2 \frac{d^2}{dx^2}, \tag{270}$$

is given by

$$\eta = \left(A_i \cos \frac{i\pi vt}{l} + B_i \sin \frac{i\pi vt}{l}\right) \sin \frac{i\pi x}{l}, \tag{271}$$

when the conditions with respect to the extreme points are unchanged. In this equation, i is any entire positive number which marks the order of the term, and A_i, B_i, are constant coefficients depending on i and on the initial state of the string. If then this state is such that η is constant only for the terms for which i is a multiple of another entire number n, the string will return to the same state at the end of each interval of time $\frac{2l}{nv}$, which is the duration of its similar and isochronous vibrations. Under this supposition, the $n - 1$ points of the curve corresponding to distances

$$x = \frac{l}{n}, \quad = \frac{2l}{n}, \quad = \frac{3l}{n}, \quad = \text{etc.},$$

will be nodes, that is, will remain at rest during the whole period of the motion.

Since the value of η is linear, every value corresponding to $i =$ 1, 2, 3, 4, etc., will be a solution, and the sum of all the values of η will also be a solution of the differential equation; hence we will have for the general integral equation

$$\eta = \Sigma_{i=1}^{i=\infty}\left(A_i \cos \frac{i\pi vt}{l} + B_i \sin \frac{i\pi vt}{l}\right) \sin \frac{i\pi x}{l}. \tag{272}$$

221. The values of $A_1 B_1$, $A_2 B_2$, etc., are in general arbitrary, and we may suppose all to vanish up to any order n, while the rest remain arbitrary. If $A_1 B_1$ are not zero, there are no actual nodes

except the fixed ends, and the first simple tone is that whose period is τ and whose wave length is $2l$. If there is one node, the period is $\frac{\tau}{2}$, and the gravest simple tone is that of wave length l; and, generally, if there are $n-1$ nodes, the period is $\frac{\tau}{n}$, and the gravest tone is the $(n-1)^{th}$ harmonic of the fundamental tone.

When the string vibrates without nodes, the series of harmonic tones is in general complete, and a practised ear can distinguish ten or more. It is also possible to make a string vibrate in such a manner that for any proposed value of n the coefficients A_nB_n, $A_{2n}B_{2n}$, etc., shall disappear, so that the component harmonic vibrations whose periods are $\frac{\tau}{n}$, $\frac{\tau}{2n}$, etc., are extinguished. When this is done, the ear does not distinguish these tones, and we may therefore conclude, from what precedes, that each component tone actually heard is produced by the corresponding harmonic vibration of the string.

222. The same general method may be applied to the longitudinal vibration of a rod, and the differential equation will be, as in the case of the longitudinal vibration of a string, of the form

$$\frac{d^2\xi}{dt^2} = V^2 \frac{d^2\xi}{dx^2}, \tag{273}$$

of which the integral equation is

$$\xi = F(x + Vt) + f(x - Vt), \tag{274}$$

and which may be put under the form of

$$\xi = x + \Sigma \cos \frac{i\pi x}{l} \left(A_i \cos \frac{i\pi Vt}{l} + B_i \sin \frac{i\pi Vt}{l} \right), \tag{275}$$

in which ξ is the distance from the fixed origin at any time t to the particles in a plane section of the rod, of which the natural distance from the end of the rod is x. The value of x therefore depends only on the particular section considered, and is independent of the origin of ξ; but if the vibrations cease, the periodic part of Eq. (275) would vanish, and we would have $\xi = x$ for all points of the rod, and therefore the periodic part gives the displacement $(\xi - x)$ at the time t of the section determined by the value of x.

The periodic part does not in general vanish for any value of x, so that there are in general no nodes. But there will be n nodes at sections for which x is any odd multiple of $\frac{l}{2n}$, provided A_i, B_i, vanish for all values of i except odd multiples of n. Thus the rod may have any number of nodes, of which those next the ends are distant from the ends by half the distance between any two consecutive nodes.

223. Differentiating Eq. (275), we have

$$\frac{d\xi}{dx} = 1 - \frac{\pi}{l}\Sigma i \sin\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi Vt}{l} + B_i \sin\frac{i\pi Vt}{l}\right); \qquad (276)$$

which, when $x = 0$ and $x = l$, becomes

$$\frac{d\xi}{dx} = 1. \qquad (277)$$

But
$$\frac{d\xi}{dx} = \frac{\rho'}{\rho},$$

in which ρ' is the natural density, and ρ is the changed density. We see, therefore, that there is no change of density at the free ends. If A_i, B_i, vanish, except where i is a multiple of n, the variable part of $\frac{d\xi}{dx}$ vanishes when x is a multiple of $\frac{l}{n}$. Hence, where there are nodes, the sections in which there is no variation of density are those which bisect the nodal intervals in the state of equilibrium, and these sections of no variation of density are also sections of greatest displacement, since, Eq. (275), $\cos\frac{i\pi x}{l}$ is equal to ± 1 for values of x which make $\sin\frac{i\pi x}{l} = 0$.

224. The vibration represented by Eq. (275) consists of an infinite number of simple harmonic vibrations, each of which might subsist by itself; the n^{th} component would have n nodes, and its period would be $\frac{2l}{nV}$, the period of the fundamental tone being $\frac{2l}{V}$; therefore the wave length is twice the length of the rod. For the general case in which there is a node at the middle of the rod, $\cos\frac{i\pi x}{l}$, vanishes for all values of i when $x = \frac{l}{2}$. Then A_i, B_i,

must vanish for all even values of i. The gravest tone is then the fundamental tone of the rod, and the higher tones of even orders disappear. The first upper tone will be a twelfth above the fundamental. In this case, the middle section might become absolutely fixed, and either half be taken away without disturbing the motion, so as to leave a rod of half the length, with one free and one fixed end. Therefore the fundamental tone of a rod with one end fixed is the same as that of a free rod of twice the length. The wave length is then four times the length of the rod, and the even orders of the harmonics are wanting.

225. The vibrations of air columns are theoretically the same as that of a free rod or one fixed at an end, and the same conclusions, modified by the elasticity of the air and its velocity of wave propagation, will theoretically apply. We will, however, determine the positions of the nodes and ventral segments of vibrating air columns in a simpler manner.

226. ***Vibrations of Air Columns.*** We will first suppose a single sonorous pulse moving in an air column, and consider, 1°, the column closed at one end and open at the other. Each stratum passes through all changes of density during the periodic time τ, while the pulse moves a distance λ; the air particles describe longitudinal vibrations, whose amplitudes depend on the intensity of the sound. When the condensation, which we suppose is in advance, reaches the closed end, the air stratum at that place, not having freedom of motion, undergoes changes of density alone. These changes are each immediately reflected in succession, and the condensation moves from the closed end with the same velocity with which it would have proceeded beyond had there been no obstruction to its progress. Hence we see that at the instant the rarefaction first reaches the closed end the reflected condensation affects the same strata as the incident rarefaction, and disregarding the loss due to incidence, the air strata will, at this instant, in the length $\frac{\lambda}{2}$ from the closed end, have their normal density throughout. The velocities of the air particles, at the same instant, will likewise be that compounded algebraically of those belonging to the reflected condensation and incident rarefaction. Now when a sonorous body is vibrating, the sound undulations follow each other

periodically, and therefore the reflected and incident pulses will be distributed throughout the column. The densities and motions of the strata will therefore result from the combination of the same elements in the incident and reflected pulses.

227. Let the curve $rm''m$A represent the direct wave at any instant, and its ordinates the corresponding compressions and dilatations of the air on the line $n'b'$ due to this wave; the curve $m''m'b$A and its ordinates will, in like manner, represent the reflected wave from the stopped end AA′.

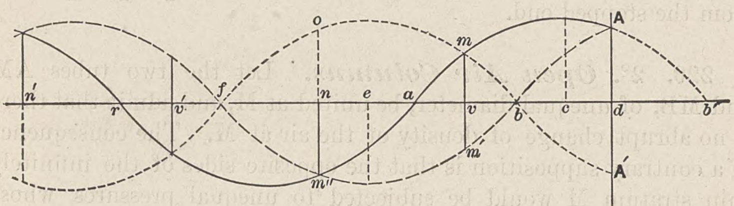

Figure 30.

We see that at points such as v, v', etc., at $\frac{1}{4}\lambda$, $\frac{3}{4}\lambda$, etc., from AA′, the condensations or dilatations due to the direct wave will always be contrary and equal to the dilatations or condensations due to the reflected wave; hence, at these points, the normal density of the air will ever exist. But at points such as n, n', etc., at distances of $\frac{1}{2}\lambda$, λ, $\frac{3}{2}\lambda$, etc., from AA′, the condensations or dilatations of each are of equal value, and of the same kind, and exist simultaneously; therefore the resultant condensation or dilatation is double that due to either. At these points then the air undergoes all variations of density during the period τ. The density at all points from n to v and to v', undergoes decreasing variations from the maximum at n to zero at v and v'.

228. With regard to the velocities of the air particles at different distances from AA′, since the motions of the particles change direction abruptly at reflection, the ordinates of the curve A′bmo will represent the velocities due to the reflected wave, and those of A$mm''r$ may now represent those of the direct wave. Then at v, v', etc., the velocities are zero only at instants separated by $\frac{\tau}{2}$, and at all other times have values that vary from zero to that represented by double the maximum ordinate; at n, n', etc., the velocities are

always zero, and therefore the air at these points is quiescent, while undergoing changes of density. At intermediate points, both changes in velocities and density occur.

Hence, we conclude that *nodes* will be developed in a column of air closed at one end, when it is traversed by a sonorous wave, at distances from the stopped end of 0, $\frac{2\lambda}{4}$, $\frac{4\lambda}{4}$, $\frac{6\lambda}{4}$, etc.

The vibrating parts between the nodes are called *ventral segments,* and their middle points are at distances of $\frac{\lambda}{4}$, $\frac{3\lambda}{4}$, $\frac{5\lambda}{4}$, etc., from the stopped end.

229. 2°. ***Open Air Columns.*** Let the two tubes AM and MB, of unequal diameter, be united at M, and admit that there is no abrupt change of density of the air at M. The consequence of a contrary supposition is that the opposite sides of the infinitely thin stratum M would be subjected to unequal pressures, whose finite difference wonld generate in M an infinite velocity in a finite time. Hence, the density has the property of continuity in its variation throughout AB. It is not essential that the variation of the velocities of the particles of air should be continuous, nor is it incompatible with this condition.

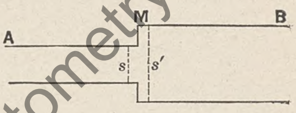

Figure 31.

Let s and s' be the areas of sections in AM and MB, indefinitely near M; v and v' the velocities of the air particles in s and s', at the time t; then $vs\,dt$ and $v's'\,dt$ will be the volume of air passing s and s' during dt, and $(vs - v's')\,dt$ will be the increment of the quantity of air in the volume ss' in the time dt, which will be proportional to the increase of density in ss'. But in order that the increment of density may be compatible with the supposed continuity of the pressure, it is evident that $(vs - v's')\,dt$ must be an infinitesimal of the second order, and equal to zero when s and s' are coincident. Hence, at the limit we have

$$vs = v's';$$

therefore, there will be a wave propagated in MB, whose intensity, determined by the value of v', will become more and more inappreciable as s' becomes greater and greater than s. Let MB be in-

creased indefinitely in area, as when the tube AM opens into the external air, then v' becomes very small, and the transmitted wave becomes negligible, as is the case in open pipes. There will then be a reflected wave in AM, composed of a rarefaction followed by a condensation, when the direct wave is a condensation followed by a rarefaction. The velocities of the air particles will then be theoretically equal in value, and the same in direction in the two waves. The curves of Fig. 30 will illustrate the case of open pipes, if $A'bof$ represent the densities and $Abm''f$ the velocities of the air particles in the reflected wave. The nodes and middle points of the ventral segments will then be at distances of

$$\frac{\lambda}{4},\ \frac{3\lambda}{4},\ \frac{5\lambda}{4},\ \frac{7\lambda}{4},\ \text{etc.,}$$

and

$$0,\ \frac{2\lambda}{4},\ \frac{4\lambda}{4},\ \frac{6\lambda}{4},\ \text{etc.,}$$

from M, the open end of the tube, respectively.

230. These laws, which determine the positions of the nodes and ventral segments of vibrating air columns, are known as Bernouilli's laws. From them we see that the harmonics of open pipes are in the order of the natural numbers, and that those of closed pipes are as the odd numbers. Thus, the open pipe can give, by an increased pressure, the octave, the twelfth, the fifteenth, etc., while the closed pipe gives the twelfth, the seventeenth, etc. Experiments with organ-pipes verify the laws of Bernouilli only approximately ; that is, that the nodes are not exactly at the positions defined above, nor are the nodes exactly places of rest. Organ-pipes are usually made to speak by forcing a current of air through a narrow slit, and causing it to impinge against a thin lip. Of the many vibratory motions produced in this manner, there is always one whose periodicity is such that, by the resonance of the pipe, its intensity will be raised to such a degree as to produce a marked and determinate musical sound, called the fundamental tone of the pipe. Other vibratory motions, which undoubtedly exist, are either destroyed by the interference of the reflected waves, or have so feeble an intensity as to be negligible. The wave length of the fundamental tone is, as we have seen above, double the length of the open pipe, or four times the length of the closed pipe, approxi-

mately. The discrepancy between experiment and theory arises from the fact that the hypothesis is not in accord with what actually occurs in the pipe. Without considering these minutely, it is sufficient to note the perturbations at the embouchure by the air current, the modifications in the pipes by the moving air, and the induced vibrations of the material of the pipe at the sides and closed end, to account for the greater discrepancies.

231. ***Relative Velocities of Sound in Different Material.*** Since in any medium, we have $\lambda = V\tau = V\frac{1}{n}$, in which n is the vibrational number for a note of definite pitch, λ the corresponding wave length in the same medium, and V the velocity of sound, it is readily seen that if free rods of different material be taken, of such lengths as to give the same note when put into longitudinal vibration, we will have

$$\lambda = V\frac{1}{n}, \qquad \lambda' = V'\frac{1}{n}, \qquad \lambda'' = V''\frac{1}{n}, \quad \text{etc.};$$

whence

$$\lambda : \lambda' : \lambda'' :: V : V' : V''.$$

But as $\frac{\lambda}{2}, \frac{\lambda'}{2}, \frac{\lambda''}{2}$, etc., are the lengths of free rods that give the fundamental tone, we see that such lengths are directly proportional to the velocities of sound in the several media, when their lengths are great compared to their cross-sections. Knowing then the velocity of sound in any material, we can by experiment find that in others by this method. Then having the velocities, we can by substitution in the formula

$$V = \sqrt{\frac{E}{D}},$$

find the value for the longitudinal modulus E.

232. Applying the same principle to any gas and comparing the velocity in it with that in air, by the formula

$$V = \sqrt{\frac{gh\Delta}{d}[1 + \alpha\theta]\gamma}, \tag{278}$$

the values of γ, or the ratio of its specific heats, can be readily obtained. By this means Dulong found the following results :

	Density.	Velocity.	$\gamma = \frac{c}{c_1}$.
Air	1.	333.	1.421
Oxygen	1.1026	317.7	1.415
Hydrogen	0.0688	1269.5	1.407
Carbon Dioxide	1.524	261.6	1.338
Carbon Monoxide, ...	0.974	337.4	1.427

Under the assumption that the gas is perfect, simple, and far from its point of liquefaction, γ is assumed to have the constant value of 1.41. The above results show that this value should be considered as the limit to which γ approximates and only reaches under the particular suppositions made.

233. ***Transversal Vibration of Elastic Rods.*** An elastic rod is a rigid body whose cross-section, considered uniform throughout, is taken as very small compared with its length. The rod or bar may be arranged in six different ways, depending on the method by which its ends are sustained, viz.:

1°. The rod may be free at both ends.
2°. It may be firmly fixed at both ends.
3°. It may be fixed at one end and free at the other.
4°. It may be supported at one end and free at the other.
5°. It may be fixed at one end and supported at the other.
6°. It may be supported at both ends.

It may yield its fundamental tone by vibrating as a whole, or give tones of higher pitch by dividing itself into vibrating parts separated by nodes. The formula

$$N = \frac{n^2 t}{2l^2} \sqrt{g \frac{E}{D}} \tag{279}$$

gives the number of vibrations in all cases, as has been verified by experiment. In this formula, N is the number of vibrations per second; n a constant depending on the manner in which the rod is arranged at the ends and on the number of nodes formed ; t is the

thickness, measured in the plane of vibration; l is the length, E the rigidity, and D the density of the rod.

234. This formula shows that the vibrational number is independent of the width, provided it be small as at first supposed; that it is directly proportional to the thickness, inversely as the square of the length, and directly as the square root of the rigidity divided by the density.

1°. The rod is free at both ends. Lissajous has determined by careful experiments that the following formulæ apply, viz.:

$$d = \frac{2l}{2n-1}, \qquad s' = \frac{5l}{2(2n-1)}, \qquad s = \frac{0.6608l}{2n-1}, \quad (280)$$

in which l is the length, n the number of nodes formed, d the distance between two consecutive nodes, s the distance from the free ends to the nearest nodes, and s' the distance from the free ends to the second nodes. Hence from these formulæ, we see that the intermediate nodes are equidistant; that the distance from the extreme nodes to the next adjacent is nearly 0.92 of the distance between two consecutive intermediate nodes; that $s : s' :: 0.2643 : 1$, and $s : d :: 0.33 : 1$. Experiment confirms these results whatever be the number of the nodes. The positions of the nodes are made visible by sprinkling sand on the bar, and noticing the lines on which it accumulates when the bar or rod is put in vibration.

2°. Both ends are fixed. When the ends are so fixed as not to modify its elasticity at these points, it can vibrate freely, and the nodes are found to be located at the same places as in a free rod of the same length, except that the extreme nodes are at the fixed ends. The first two of formulæ (280) are then applicable to this case.

3°. The rod is fixed at one end and free at the other. There will then be 0, 1, 2, 3, ... nodes depending upon the manner by which it is put in vibration. If the fixed end be regarded as a node, the first of the above formulæ is applicable, and the other two apply to the free end only. Therefore these first three cases are all reducible with the modifications mentioned to that of a free rod at both ends.

4° and 5°. In these cases the supported end may be considered as an intermediate node, and we can consider the rod as half of a rod of double the length, free or fixed at both ends in which the

number of nodes is $2n-1$. Replacing l by $2l$ and n by $2n-1$, we then have

$$d = \frac{4l}{4n-3}, \qquad s' = \frac{5l}{4n-3}, \qquad s = \frac{1.3216\, l}{4n-3}, \tag{281}$$

of which the last two apply only to the case where one of the ends is free.

6°. If the supported ends be regarded as intermediate nodes we have

$$d = \frac{l}{n-1}. \tag{282}$$

235. ***Harmonic Vibrations of Elastic Rods.*** When the vibrating parts are known, the harmonies of the rod are easily determined, and considering the fixed extremities as nodes, the formulæ of Lissajous above given become general for the six cases. In the first three cases the sounds resulting are the same for the same number of nodes, whatever be the condition of the extremity, whether fixed or free. The numbers of vibrations are as $3^2, 5^2, 7^2, \ldots. (2n-1)^2$, when there are $2, 3, 4, \ldots. n$ nodes. In the 4° and 5° cases where one of the extremities is supported, the vibrational numbers are as $5^2, 9^2, 13^2, \ldots. (4n-3)^2$; and in the 6° case the numbers are $1^2, 2^2, 3^2, \ldots. (n-1)^2$, n being the number of nodes. Comparing in all these cases the vibrational numbers for two nodes, we have

$$9 : \frac{25}{4} : 4, \qquad \text{or} \qquad \frac{9}{4} : \frac{25}{16} : 1;$$

and therefore generally

$$\frac{(2n-1)^2}{4} : \frac{(4n-3)^2}{16} : (n-1)^2.$$

For d we have

$$\frac{2l}{2n-1}, \qquad \frac{4l}{4n-3}, \qquad \frac{l}{n-1}.$$

Substituting the value of n taken from the latter in the former, we have

$$N = \frac{l^2}{d^2}. \tag{283}$$

If $d = l$, which corresponds to a rod supported at both ends and yielding its fundamental sound, we have $N = 1$. We therefore con-

clude that when a rod gives a harmonic, the parts comprised between the nodes vibrate as rods whose extremities are supported and whose length is the distance between the nodes, and that the vibrational number is inversely as the square of this length. This conclusion is inapplicable to the first nodes, because they are more or less influenced by the extremities.

236. ***Tuning Forks.*** A tuning fork may be regarded as a rod or bar free at both ends. Experiment shows that in proportion as a bar free at both ends is bent or curved the extreme nodes approach each other. Thus, in the figure the bar ab if supported at the points 1, 2, one fourth the length of the bar from the extremes, will when vibrated transversely develop nodes at these points. In the forms $a'b'$, $a''b''$, $a'''b'''$, the length remaining unchanged, the nodes approach each other as indicated in the figure.

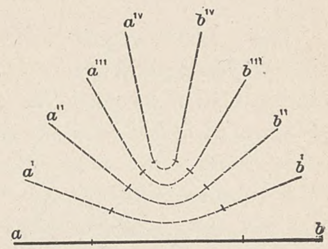

Figure 32.

The laws which govern the vibration of a fork whose section is rectangular have been experimentally found to be; 1°, that the vibrational number is independent of the width; 2°, proportional to the thickness; 3°, inversely proportional to the square of the length increased slightly. The length is taken as equal to the projection of the prongs on the medial line of the fork. For a fork of rectangular cross-section we have from the experiments of Mercadier.

$$N = \kappa \frac{t}{2l^2(1.012)^2} \tag{284}$$

in which N is the vibrational number, t the thickness, and l the length; κ is a constant which for steel is found to be 818270. When the fork yields its fundamental note its method of division is shown in the figure. The overtones of a fork correspond to vibrational numbers which are to each other, beginning with the first; as, $3^2 : 5^2 : 7^2$: etc. The vibrational number of the first overtone is about $\frac{25}{4}$

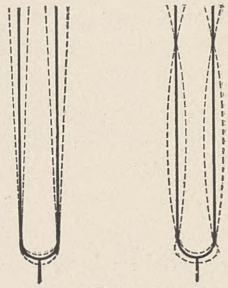
Figure 33.

that of the fundamental. Helmholtz found by experimenting on many forks, that it varied from 5.8 to 6.6 that of the fundamental. These overtones are so high, that they are generally of short duration, and they are also inharmonic with the prime. Tuning forks are generally mounted on their resonant boxes, by which arrangement the prime tone of the fork is greatly reinforced to the disadvantage of the overtones. The duration of the vibration of a fork although theoretically constant, is found to increase slightly with an increase of amplitude and temperature, thus slightly lowering the pitch. This is, however, not appreciable to the hearing, but can be detected by any of the graphical methods for determining the number of vibrations in a given period. It is a matter of importance in determining the initial velocity of projectiles, by means of the Schultz chronoscope or other devices, where the vibrations of a tuning fork enter into the calculation, to limit the amplitude and to take note of the temperature, in order to obtain uniform and reliable results. When the amplitude does not surpass 3 or 4 mm. and the temperature varies but little beyond the ordinary atmospheric temperature, the vibrational number may be taken as constant within .0001 of its value.

237. ***Vibration of Plates.*** Plates are rigid bodies, generally of metal or glass, whose length and breadth are very great compared with their thickness. To put them in vibration, one or more points are fixed and a violin bow is drawn across an edge. The circumstances of vibration are exhibited by sprinkling fine sand over the surface and examining the nodal lines formed by the sand which seeks that part of the plate which is at rest. The parts of the plate separated by a nodal line, evidently vibrate in opposite directions, and therefore for permanent figures the number of vibrating parts must be even. When the plate yields its fundamental tone the resulting figure is the simplest that can be formed, and as the plate separates into a greater number of vibrating parts, the figures become more complex. Chladni has given to these figures the name of *Acoustic figures.* As yet, from the inherent difficulties of the problem, the mathematical laws have not been deduced, but experiment has assigned the following as the laws of vibrating plates, viz.; 1°, the vibrational numbers of plates of the same form and of the same material are inversely as the squares of

the homologous dimensions; 2°, and are proportional to the thickness. Hence we have

$$n : n' :: \frac{t}{l^2} : \frac{t'}{l'^2}.$$

If a rectangular plate be so constructed that a system of nodal lines parallel to the length be formed by a sound, which gives another system of nodal lines parallel to the breadth, when it is vibrated in these two ways, then if at any of the middle points of the ventral segments it be vibrated so as to produce the same sound, these two systems will simultaneously exist and the acoustic figure will result from the combination of these two systems. The figure illustrates five such plates where the numbers of the nodal lines are in the ratios of 2:3, 2:4, 3:4, 3:5, 4:5. Other combinations illustrative of the vibrations of plates are reserved for the lectures.

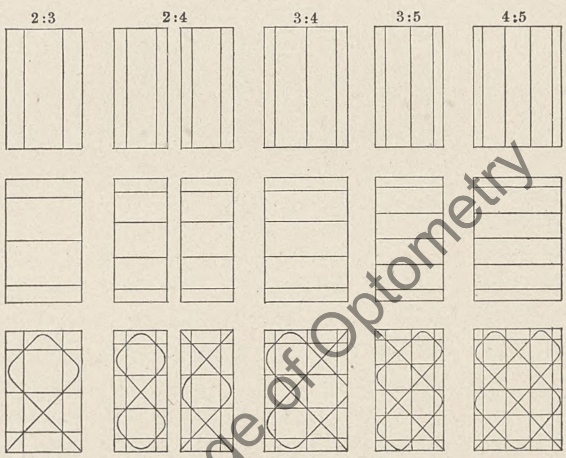

Figure 34.

238. ***Vibration of Membranes.*** When a stretched membrane is near a sounding body, the air transmits to it the vibratory motion. It can respond, however, only to certain sounds depending on its tension, and thus enter into synchronous vibration. This fact is made evident by the acoustic pendulum, or by the nodal lines formed by sand sprinkled upon it, as in the case of the

vibration of plates. The frames upon which the membranes are stretched are generally square or circular. Experiment has confirmed the following deductions of Poisson and Lamé, with respect to the vibrations of square membranes, viz.:

1°. Membranes respond only to certain sounds, separated by determinate intervals.

2°. To each sound a system of nodal lines corresponds, parallel to the sides of the membrane, and whose numbers are represented by n and n'.

3°. The nodal lines which correspond to the same sound form a system of figures, such that we can pass from one to the other by continuous changes in varying the mode of disturbance, without changing the sound; but we can never pass in a continuous manner from the lines of one sound to those of another.

Circular membranes can only give nodal lines along the diameters or circumferences, either separate or combined, depending on the method of vibration and on the point or points of enforced rest.

239. Because of the limited time allotted to this part of the course, many subjects of importance are necessarily omitted in the text. Among these are,

1°. The theory of beats, and resultant sounds.

2°. The phenomena of interference, whose consequences, however, are readily derived from the discussion in Arts. 65–68.

3°. The graphical and optical methods of the study of sonorous vibrations, and that by sensitive and manometric flames.

4°. The phenomena of vibrations of air columns in organ-pipes, of elastic rods, of plates and membranes, with the applications of the latter in the phonograph, phonautograph, and telephone.

By means, however, of a very complete acoustical apparatus, mainly from the workshop of Koenig, the celebrated physicist of Paris, the omitted parts, as well as those treated of above, are illustrated in the lectures, which largely supplement and complete the study of the text.

240. The nature and essential principles of undulatory motion, as illustrated by sonorous vibrations, have received sufficient attention to enable the student to prosecute understandingly the study of similar principles connected with light in the analogous subject of optics.

PART III.

OPTICS.

241. Light is the agent by which the existence of bodies is made known to us through the sense of sight.

That branch of physical science which treats of the properties of light and the laws of its transmission is called ***Optics.***

242. It is divided into two parts:

1°. ***Geometrical Optics,*** which embraces all the phenomena relating to the propagation of rays, based on certain experimental laws, and which is entirely independent of any theory as to the nature of the luminous agent.

Experiments in Geometrical Optics, however carefully made, can never accurately prove the laws of light propagation, but serve merely to establish a certain degree of probability of their truth, and which, when applied to other phenomena of the same nature, strengthen this probability in proportion as the application is more extended.

2°. ***Physical Optics,*** which is based on the theory of undulations, and seeks to explain by this theory the nature of light, and of all the phenomena arising from the action of rays on each other.

243. That light is not a material substance, but is merely a *process* going on in some medium, is proved by the phenomena of interference, in which results of various magnitudes occur, from less to greater, or the reverse, depending upon the manner in which the interference takes place, even when the combining magnitudes are themselves constant in value.

244. The undulatory theory asserts that light is due to the transmission of energy from luminous bodies to the finely-divided parts of the optic nerve, spread over the interior concave surface of

the eye. This energy is conveyed by the optic nerve to the brain, and there transformed into the sensation of sight.

The transmission of the energy is accomplished by undulatory motion in a medium called the *luminiferous ether*. There is no direct proof of the actual existence of the ether, and its assumption can only be regarded as an extremely probable hypothesis, supported by nearly all the known phenomena of light, and directly contradicted by none.

Within the present century, its reality has been almost universally accepted, and as a consequence the undulatory theory has entirely supplanted the rival hypothesis of the materiality of light molecules, known as the emission theory, which had, however, held its ground for many years.

245. The accepted properties of the luminiferous ether have resulted from theoretical considerations, modified from time to time by deductions from experimental observations, and while there are several imperfections yet to be removed, nevertheless the strong array of unquestioned facts, both observed and predicted, has established these properties as a satisfactory foundation upon which modern physical optics is now constructed.

The luminiferous ether is considered to be a material substance of a more rare and subtile nature than the ordinary matter affecting the senses, and to exist not only within these bodies, but throughout space. It has great elasticity, and is capable therefore of transmitting its particular energy over vast distances, with great velocity and with inappreciable loss. That this energy is not transmitted instantaneously has been proved by direct experiment, and concluded from several astronomical observations.

246. That light is propagated in right lines from the source is a fact of observation and experiment. This statement, however, while absolutely true, is subject to modification when taken in the ordinary sense of the language. Thus, we have seen that while sound is propagated in right lines from its source, it is capable of spreading around an obstacle, so that sound can be heard out of the direct line of the source ; so, in a less degree, we can see around an obstacle, as will be shown in the discussion of the diffraction of light.

The acoustic shadow, however, is as much less marked than the optical shadow as the wave lengths of sound are greater than the

wave lengths of light. But for the explanation of the principles of geometrical optics it is unnecessary to consider this refinement.

247. Bodies are called *self-luminous* when they are themselves the sources of light, and rays proceed directly from them. They are visible because of their emanating rays. Other bodies are called *non-luminous*, and become visible because rays from luminous bodies are reflected from their surfaces.

A luminous point, or origin of light, is a very small portion of a luminous surface. When light emanates from a luminous point, we consider it made up of *rays* of light, each of which is the smallest portion of light which can be transmitted. The ray is the right line along which the undulation is propagated, and is practically a mere conception, indicating direction.

A collection of parallel, diverging, or converging luminous rays is called a *beam* of light, and sometimes a *pencil* of light, the latter name being generally applied to the last two cases.

The *axis* of a beam is the geometrical axis of the cylinder or cone of rays; when the axis is normal to the deviating surface, the beam is *direct*, and when inclined to it, *oblique*.

248. When a beam of light is incident upon any surface, it is generally separated into three portions, viz., a part is scattered or diffused over the surface, by which the surface becomes visible, a second part is reflected, and the remainder is refracted.

The proportion of the several parts depends on the polish of the surface, the angle of incidence, and the nature of the medium. A perfectly polished surface would be invisible, and the incident beam would be separated into a reflected and refracted beam alone; of course, such a polish is not practicable. Light regularly reflected has its intensity increased with the degree of polish, while the intensity of irregularly reflected light is similarly diminished. The intensity of regularly reflected light from the surface of water is, at the incidences of

	0°,	40°,	60°,	80°,	$89\frac{1}{2}$°,
about	1.8%,	2.2%,	6.5%,	33%,	72%.

At normal incidence, water, glass, and mercury reflect 1.8%, 2.5%, and $66\frac{2}{3}$%, respectively. The differences at small angles of incidence are more marked than at greater angles, since while both

water and mercury reflect the same at $89\frac{1}{2}°$, the former reflects but $\frac{1}{36}$ as much as the latter at normal incidence.

249. A *medium* is any substance which permits the passage of light through it.

Since the luminiferous ether is supposed to pervade all matter, it might be inferred that all bodies could be classed under the head of media for light. Gold, although one of the most dense of substances, does permit the passage of light, when beaten into a very thin leaf; and no doubt if other opaque bodies possessed an equal malleability, the same property would belong to them.

But owing to internal reflection and consequent interference, it is assumed that an inappreciable quantity of light, if any, passes through very small thicknesses of *opaque* bodies. Glass, air, water, and all other matter which permit the passage of light freely, are said to be transparent. *Translucency* is a term applied to such bodies as permit the passage of diffused light; thus, ground glass and flint are translucent, while clear glass and quartz crystal are transparent.

250. Since light is assumed to result from undulatory motion in the luminiferous ether, all the consequences deduced in the discussion of the properties of this kind of motion in Part I are at once applicable to the phenomena of light.

251. ***Shadows and Shade.*** From each luminous point considered as an origin of disturbance, undulations proceed along right lines in all directions from this origin. Therefore, whenever they meet an opaque body, this undulation will be deviated from its original direction, and the effect of light will be wanting along this direction prolonged.

The absence of this effect is called the *shadow* of the point of the opaque body.

The line of the surface of the opaque body, along which rays drawn from the luminous point are tangent, is called the *line of shade.* Since each point of the luminous surface is an origin of light, we see that in all actual cases the shadow of an opaque body must be indistinct near its boundary, and gradually merge into the illuminated surface surrounding the shadow, whenever the luminous source is of an appreciable area. This modified portion of the shadow is due to the overlapping of the cones of rays proceeding

from each luminous point, and is called the *penumbra*. It is limited by the space between the two cones, whose elements are tangent to the luminous surface and the opaque body, one having its vertex between the two, and the other its vertex on the further side of either one of the surfaces. The softness of shadows in general is due to the finite extent of luminous surfaces.

252. Every point of the luminous source emitting rays in all directions, each will carry an image of its luminous point.

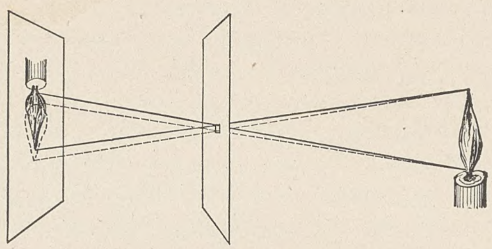

Figure 35.

Thus, if a lighted candle be placed in front of a small aperture of a darkened chamber, the aperture will permit the passage of a limited number of the rays from every point of the candle, each ray, however, carrying an image of its radiant. The image, as shown in Figure 35, will be inverted.

If another aperture be made near the first, a second image of the candle will be formed, overlapping the first, and, while the luminosity will be increased, the image will lose distinctness, because of this overlapping. The diffused light of a room during the day is due to the overlapping images of external objects, caused by rays proceeding from each of them, thus making their individual images indistinct. A small aperture in a darkened room will permit the formation of an inverted image of the external scenery upon a screen placed within the room near the aperture.

253. ***Photometry.*** The eye possesses the property of distinguishing color and intensity.

In determining variations of intensity, the judgment is only approximate when the colors are the same, and the difficulty of this appreciation is increased when the colors differ. Equality of in-

tensity can readily be determined by the eye, while it is not possible to ascertain the numerical ratio of different intensities by direct observation.

Photometry has for its object the measurement and comparison of the intensities of different lights.

254. The principle of all photometric methods is to arrive at this comparison, by the appreciation of the equality of illumination of two near surfaces, physically identical. In assuming the distance of the luminous source from the illuminated surface to be great in comparison with the dimensions of the surface, and remembering that the intensity of the light is due to the molecular kinetic energy, we readily see, if there be no absorption of this energy during transmission through the intervening media,

1°. That the intensity of the illumination on the unit area of any surface, taken normal to the direction of propagation, at a distance d from the luminous source, varies as $\frac{1}{d^2}$.

2°. That if I represent the intensity of any given light, and if it be supposed to illuminate uniformly any area A, the intensity on a unit of area varies as $\frac{1}{A}$.

3°. That the quantity of light emanating from any luminous element, and hence the intensity of illumination on the unit area, is proportional to the cosine of the angle made by the normal to the element with the direction considered, and hence varies as the cosine of the inclination, or $\cos i$.

4°. That if the area on which the light falls is inclined to the direct line of propagation, the illumination on the unit area is proportional to the cosine of the angle made by this line and the normal to the surface, or to $\cos i'$.

5°. That the illumination on the unit area will vary with the *intrinsic brightness* of the source. The intensity of the illumination on the unit area, parallel to the source, at the distance unity, may be taken as the measure of the intrinsic brightness.

255. Let S and S' be the projections of the luminous and the illuminated surfaces, respectively, on a plane normal to the direction of the luminous rays; B the intrinsic brightness of the

source ; d the distance apart of the two surfaces, and I the intensity of the illumination; then, from the above principles, we have

$$I = B\frac{SS'}{d^2}. \tag{285}$$

Making $S' = 1$, and calling $I_,$ the total brilliancy of the source at the distance d, we have

$$I_, = B\frac{S}{d^2}. \tag{286}$$

$\frac{S}{d^2}$ is the apparent area of the source seen from the illuminated surface, and making this equal to unity, we have

$$I_{,,} = B. \tag{287}$$

Therefore the intrinsic brightness of the source is the total brilliancy of the apparent unit of area of the luminous surface at the distance 1.

The general method of comparison of the intrinsic brightness of two sources consists in permitting the rays from each source to fall, nearly normal, upon adjacent portions of the same surface; then to increase the distance of the stronger light, until the eye judges the illumination to be equal. We then have

$$\frac{BS}{d^2} = \frac{B_,S_,}{d_,^2}; \tag{288}$$

from which, by substituting the known values of d, $d_,$, S and $S_,$, the ratio of $B_,$ to B can be determined.

256. The *apparent* intrinsic brightness of an object is equal to the quantity of light received from it by the eye, divided by the area of the picture on the retina. Therefore, since the apparent illumination of the object is $B\frac{S}{d^2}$, and the area of the retinal picture is $\frac{S}{d^2}$, the apparent intrinsic brightness will vary with the real intrinsic brightness B, and the object will appear equally bright at all distances.

This result is deduced under the supposition that no light from the object is absorbed by the medium through which it passes, and is therefore only an approximation.

257. *Velocity of Light.* In 1675, the Danish astronomer, Rœmer, noticed certain discrepancies with regard to the observed times of the eclipses of Jupiter's satellites, which he correctly attributed to the finite velocity of light.

To show this, let S be the sun, EE′ the earth's orbit, JJ′ the orbit of Jupiter, and *ss′* the orbit of Jupiter's inner satellite.

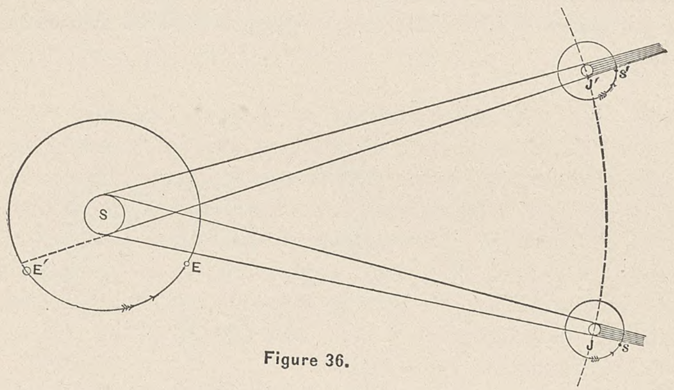

Figure 36.

The planets and satellites shine by the reflected light of the sun, and therefore cast shadows, whose axes are on the right lines joining their centres with the centre of the sun. Because of the position of the orbit of the satellite with respect to the plane of Jupiter's orbit, the satellite enters Jupiter's shadow at every revolution, and is eclipsed. If light traversed space instantaneously, its entrance into and exit out of the shadow might be noted at the exact instants at which these phenomena occurred, independently of the relative positions of the earth and Jupiter.

But when Jupiter is near opposition, as at J, the interval between two successive disappearances of the satellite in entering, or between two successive reappearances on emerging from the shadow is found to be about 42 hr. 30 min. The periodic time of Jupiter being about 11 yr. 10 mo., he advances but a short distance, as to J′, while the earth moves to E′ near conjunction.

Their distance is now increased by very nearly that of the diameter of the earth's orbit, and the times of apparent immersion of the satellite are delayed beyond the computed times by about 16 min. 26 sec. Since the periodic time of the satellite is constant, Rœmer

therefore concluded that light required 16 min. 26 sec. to traverse this diameter.

If this diameter were accurately known, and the exact instant of the eclipse could be noted, a very nearly exact measure for the velocity of light could be computed. The reduction of more than a thousand eclipses of Jupiter's satellites, by Delambre, gave 473.2 mean solar seconds for the time of travel, which corresponds to a solar parallax of 8.878″, and to a velocity of 298,793 kilometres per second.

258. ***By the Aberration of Light.*** Bradley, in 1728, accounted for the aberration of the fixed stars by assuming that the velocity of the earth's orbital motion had an appreciable ratio to the velocity of light. By assuming an ideal star at the pole of the ecliptic, the value of the constant of aberration, according to his determination, is 20.25″, which corresponds to a solar parallax of 8.881″. According to W. Struve, this constant should be 20.445″, decreasing the parallax to 8.797″, and corresponding to a velocity of 296,067 kilometres per second.

The principle on which this method is based is given in the text on Astronomy.

259. ***By Actual Measurement.*** Owing to the great velocity of light, it is not possible to measure directly the very small interval of time required for light to traverse any terrestrial distance. But Fizeau, Foucault, Wheatstone, Cornu, and more recently Michelson, have succeeded in obtaining its value within very near limits. The essential principle of the experiment by Fizeau consists in causing a toothed wheel to revolve with great, but uniform velocity, in a plane perpendicular to the track of a small parallel beam of light. The toothed wheel in its rotation alternately permits and obstructs the passage of the beam, according as an interval or a tooth is interposed in its track. The beam of light, after traversing the distance determined upon, is reflected by a small mirror, and may or may not be intercepted on its return, depending on the ratio of the velocity of rotation of the wheel and the velocity of light. Should the velocity of rotation be such that the returning beam passes through the next interval, the circumference of the wheel would have moved through an angle equal to

that subtended by a tooth and an interval, while the light has traversed double the distance from the wheel to the reflector.

When the angular velocity of the wheel is doubled, the light passes through the second interval, and so on. The value for the velocity of light determined by this method is 315,364 kilometres. Cornu has recently made use of the same method, but with a very much improved apparatus, and has found, as the mean of 504 experiments, the value of 300,400 kilometres for the velocity of light in vacuo, with a probable error of less than .001.

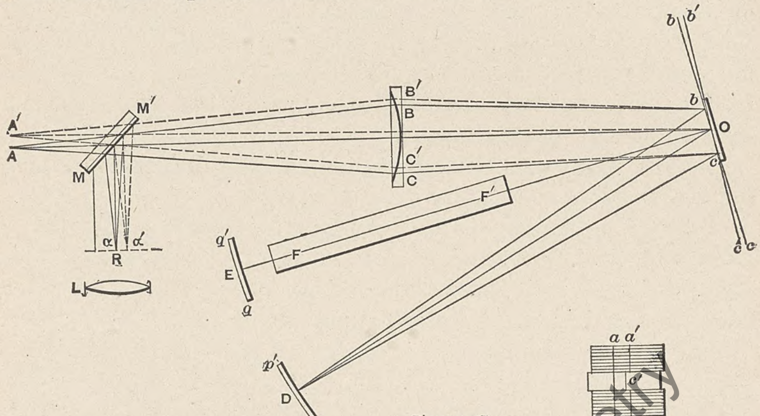

Figure 37

260. Foucault's method is a modification of the preceding. Let A, in Figure 37, be a luminous line, BC a lens whose focal length for the position A is Bb + bD, bOc a revolving mirror, D and E circular mirrors whose centre is at O, MM' a glass plate, R a reticle, and L an eye lens to view the image of A. Now if the mirror O is at rest, the path of a ray from A, passing through the lens BC and reflected from O, is ABbD; returning by reflection from D, its path is DcCA. A part of its light is reflected from the first surface of MM', and the image of A is seen coincident with its object at α. If now the mirror O is put into sufficiently rapid rotation, the returning ray meets it at b'Oc', and the ray is reflected along cC'A', and its image is seen at α'. The angle bOb' is known from the velocity of rotation, the distance OD is given, and the displacement $\alpha\alpha'$ is measured by a micrometer.

These data serve to measure the velocity of light in terms of the

angular velocity of O. By the addition of a tube filled with water at FF′, the velocity of light in water was found and shown to be less than that in air.

In the diagram annexed to the figure, *ab* is the position of the image when O is at rest, *c*′ when O has a determinate velocity, and *a*′*b*′ the corresponding position of the image after the ray has traversed the water. The result of this determination is 298,187 kilometres for the velocity of light, corresponding to a solar parallax of 8.86″. Michelson, by an ingenious modification of the method of Foucault, by which he separated his mirrors 2000 feet, and caused one of them to revolve 257 times per second, obtained a deflection of his image exceeding 133 millimetres, and thus obtained results which are claimed to be exact to within one ten-thousandth, due to this element of deflection.

As the mean of 1000 observations, he has determined 299,930 kilometres per second for the velocity of light in vacuo.

A new investigation of this important constant, under the direction of Prof. Newcomb, is now in progress, and which, when completed, will undoubtedly be as close an approximation to the true value as the present state of experimental science can furnish.

261. Assuming that light is due to the transversal vibrations of the luminiferous ether, we see, Eq. (119), that in isotropic media the velocity of light depends on the coefficients *a*, *b*, *c*, etc., which are functions of the elasticity and density of the medium.

In homogeneous light, or that in which λ is constant, V will therefore vary when light passes from one medium into another. The conclusions derived by supposing a variation in λ, the medium remaining the same, will be considered under the dispersion of light.

GEOMETRICAL OPTICS.

262. In geometrical optics it is only necessary to take account of the variation of the velocity due to a change in the elasticity and density of the ether, in passing from one isotropic medium into another. Hence, we consider homogeneous light alone in the discussions which follow. These changes are given by the formula

$$\sin \phi = \mu \sin \phi' = \frac{V}{V'} \sin \phi'. \qquad (289)$$

The ratio μ is called the *index of refraction;* it is the ratio of the velocity of light propagation in the two media, and is called the *absolute* index when the medium from which it passes is the ether. When any two other velocities are compared, the ratio is called the *relative* index; the relative index is then only the ratio of the two absolute indices. When reflection is considered as a particular case of refraction, μ is always taken as -1.

263. A *radiant* is a point from which the rays proceed; it is said to be *real* when the beam is parallel or diverging, and *virtual* when converging. A *focus* is the point in which the rays meet after deviation, or in which they would meet if prolonged in either direction; in the former case the focus is real, and in the latter, if the point of meeting is found by prolonging the rays backward, it is virtual. A radiant and its focus are the centres of curvature of the undeviated and deviated pencils, respectively. In the following discussions, distances estimated in the direction of wave propagation, from any origin whatever, are taken as *negative,* and in the contrary direction as *positive.*

264. ***Deviation of Light by Plane Surfaces.*** Let us suppose the incident medium to be any whatever, as air, and that the ray enters any other medium, as glass, whose surface is plane, Then, Figure 38, we have, for the first refraction,

$$\sin\phi = \mu \sin\phi', \tag{290}$$

in which μ is the relative index of air referred to glass; and for the first reflection we have

$$\sin\phi = -\sin\phi. \tag{291}$$

The angle ϕ' is less than ϕ, because $\mu = \frac{V}{V_{\prime}}$ is greater than unity, since the velocity of wave propagation of light in air is found by experiment to be greater than that in glass. Should the velocity in the medium of intromittance be greater than that in the medium of incidence, μ would be less than unity and ϕ' would be greater than ϕ. The re-

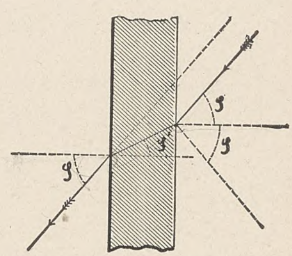

Figure 38.

fracted ray will meet the second bounding surface of the medium, which we suppose plane, and its direction on emergence into air is obtained from the equation

$$\sin \phi_{,} = \frac{V_{,}}{V} \sin \phi'' = \frac{1}{\mu} \sin \phi''. \tag{292}$$

If the second plane surface is parallel to the first, then $\phi_{,}$ in the last equation will be equal to ϕ' in Eq. (290), and we will have $\phi'' = \phi$; whence we see that the ray is not ultimately deviated in passing through a medium bounded by parallel plane surfaces, and emerging into the incident medium.

The refracted ray is, as shown by the figure, not coincident with the incident ray, but is displaced, parallel to it, by a distance depending upon the thickness of the plate and the angle of incidence ϕ. From Eq. (292) we also see that the refractive index of one medium referred to another is the reciprocal of the refractive index of the latter referred to the former.

265. The parallelism of emergent and incident rays is likewise true when light traverses any number of parallel plates of different refractive media. Thus, let A, B, C, be consecutive parallel plates of different media, and μ, μ', μ'', their respective absolute refractive indices; then, in supposing incident light to proceed from and to emerge finally into a vacuum through these plates, we will have

$$\left.\begin{aligned}
\sin \phi &= \frac{V}{V_{,}} \sin \phi' = \mu \sin \phi', \\
\sin \phi' &= \frac{V_{,}}{V_{,,}} \sin \phi'' = \frac{\dfrac{V_{,}}{V}}{\dfrac{V_{,,}}{V}} \sin \phi'' = \frac{\mu'}{\mu} \sin \phi'', \\
\sin \phi'' &= \frac{V_{,,}}{V_{,,,}} \sin \phi''' = \frac{\dfrac{V_{,,}}{V}}{\dfrac{V_{,,,}}{V}} \sin \phi''' = \frac{\mu''}{\mu'} \sin \phi''', \\
\sin \phi''' &= \frac{V_{,,,}}{V} \sin \phi^{\text{iv}} = \frac{1}{\mu''} \sin \phi^{\text{iv}};
\end{aligned}\right\} \tag{293}$$

whence we have,

$$\sin\phi = \mu' \sin\phi'' = \mu'' \sin\phi''' = \sin\phi^{\text{iv}}. \tag{294}$$

Experiment shows that the relative index for solids and liquids, *referred to air,* differs but little from their absolute indices, and in geometrical optics, these relative indices may be assumed for the absolute indices, when air is the incident medium.

266. ***Refraction by Optical Prisms.*** An optical prism is a refractive medium bounded by two plane surfaces enclosing a diedral angle; the faces of this angle form the deviating surface for luminous rays. In the optical prism BAC take the plane of incidence normal to the axis of the prism.

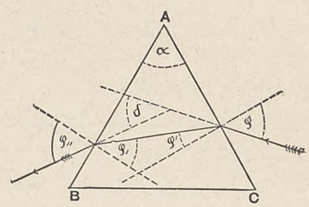

Figure 39.

Let α be the refracting angle of the prism, ϕ, ϕ', and $\phi_{,}$, $\phi_{,,}$, the first angles of incidence and refraction, and second angles of incidence and refraction, respectively; let μ be the relative index, and δ the total deviation of the ray after passing through the prism. Then we have

$$\left.\begin{aligned} \sin\phi &= \mu\sin\phi', \\ \sin\phi_{,} &= \frac{1}{\mu}\sin\phi_{,,}; \end{aligned}\right\} \tag{295}$$

$$\left.\begin{aligned} \phi' + \phi_{,} &= \alpha, \\ \phi + \phi_{,,} &= \alpha + \delta, \\ \therefore\quad \delta &= (\phi - \phi') + (\phi_{,,} - \phi_{,}). \end{aligned}\right\} \tag{296}$$

δ is therefore a function of but one independent variable. Let this be ϕ'; then we have, from Eqs. (296), by differentiation,

$$\frac{d\phi_{,}}{d\phi'} = -1, \tag{297}$$

$$\frac{d\delta}{d\phi'} = \frac{d\phi}{d\phi'} + \frac{d\phi_{,,}}{d\phi_{,}}\cdot\frac{d\phi_{,}}{d\phi'} = \frac{\mu\cos\phi'}{\cos\phi} - \frac{\mu\cos\phi_{,}}{\cos\phi_{,,}}, \tag{298}$$

which, when placed equal to zero, gives $\phi' = \phi$, and $\phi = \phi_{\prime\prime}$. Taking the second differential coefficients, we have

$$\left.\begin{aligned} \frac{d^2\delta}{d\phi'^2} &= \frac{d^2\phi}{d\phi'^2} + \frac{d^2\phi_{\prime\prime}}{d\phi_{\prime}^2}\cdot\frac{d\phi_{\prime}^2}{d\phi'^2} \\ = \mu\Big(&\frac{\mu\cos^2\phi'\sin\phi - \cos^2\phi\sin\phi'}{\cos^3\phi} \\ &+ \frac{\mu\cos^2\phi_{\prime}\sin\phi_{\prime\prime} - \cos^2\phi_{\prime\prime}\sin\phi_{\prime}}{\cos^3\phi_{\prime\prime}}\Big). \end{aligned}\right\} \quad (299)$$

Substituting the above values, we get

$$\frac{d^2\delta}{d\phi'^2} = \frac{2\mu(\mu\cos^2\phi'\sin\phi - \cos^2\phi\sin\phi')}{\cos^3\phi}, \quad (300)$$

which is positive when $\mu > 1$, and negative when $\mu < 1$. The former, therefore, corresponds to a minimum and the latter to a maximum. Therefore we conclude, that when the angle of incidence is equal to the angle of emergence, the deviation is a minimum when the medium of the prism has a refractive index greater than unity, and the deviation is a maximum when the refractive index is less than unity.

The former is the more usual case, and will hereafter be assumed.

Substituting this value of $\phi_{\prime\prime}$ in Eq. (296), we have

$$\phi = \frac{\delta + \alpha}{2}, \quad \text{and also} \quad \phi' = \frac{\alpha}{2}.$$

This furnishes a simple method of determining experimentally the relative index of any medium referred to air. Thus, if the medium be a solid, a prism of it is formed whose refracting angle and the position of a small parallel beam of homogeneous light at its minimum deviation are experimentally determined. The values of α and δ, substituted in the equation

$$\sin\left(\frac{\delta + \alpha}{2}\right) = \mu\sin\frac{\alpha}{2}, \quad (301)$$

enables us readily to find μ.

If the substance be a liquid, a hollow prism whose sides are plates of glass, each with parallel faces, is filled with the liquid, and the same measurements made as in the case of the solid prism.

267. The refractive indices for the same medium vary with the wave length, since, as shown by Eq. (119), the velocity varies with this length. Therefore, the color of the ray must be stated for which the refractive index is taken.

TABLE OF REFRACTIVE INDICES FOR YELLOW RAYS.

Solids.		*Liquids.*	
Chromate of Lead,	2.50 to 2.97	Carbon Bisulphide, .	1.678
Diamond, . . .	2.47 to 2.75	Olive Oil, ...	1.470
Carbonate of Lead,	1.81 to 2.08	Sulphuric Acid, . .	1.429
Flint Glass, . . .	1.57 to 1.58	Hydrochloric Acid, .	1.410
Crown Glass, . .	1.525 to 1.534	Alcohol, ...	1.372
Plate Glass, . . .	1.514 to 1.542	Water, ...	1.336
Ice, ...	1.310		

268. ***Deviation of Light by Spherical Surfaces.*** Let O be any radiant, BAD any spherical reflecting surface, and C its centre of curvature; then the direction of any ray deviated by reflection can be readily determined when ϕ is known. Let OI be any incident ray and If' be the corresponding reflected ray; then all incident rays which proceed from the same radiant, and

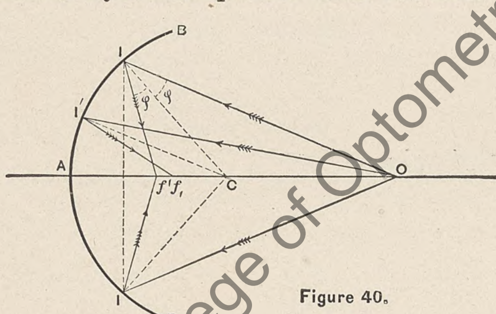

Figure 40.

make the same angle with the axis OA, will after reflection meet in f'. As this point of incidence I is taken nearer to A, the corresponding point f' will move to the right and approach some limiting position on the axis, as $f_{,}$, and the pencil may be taken so small as to make the distance $f'f_{,}$ less than any assignable value. The position $f_{,}$ is called the *geometrical* focus of the reflected pencil. We can approximate in the same way to the geometrical focus of a

pencil refracted by a medium bounded by a spherical surface, when ϕ and μ are known.

The *principal focus* of a spherical reflecting or refracting surface is the geometrical focus of a small *parallel* beam of light, whose axis coincides with the axis of the surface.

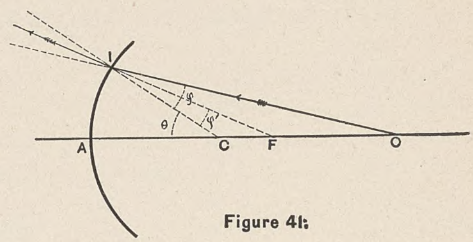

Figure 41:

Let F be the virtual focus of the rays which proceeding from O, Fig. 41, meet the surface at the angular distance θ from the axis, after deviation by refraction; then we have, in the triangles CIO and CIF,

$$\frac{\sin \phi}{\sin \theta} = \frac{f - r}{s}, \qquad \frac{\sin \theta}{\sin \phi'} = \frac{u}{f' - r}, \tag{302}$$

in which $f = \text{AO}$, $f' = \text{AF}$, $r = \text{AC}$, $u = \text{FI}$, and $s = \text{OI}$.

Eliminating θ, and combining the resulting equations with

$$\sin \phi = \mu \sin \phi',$$

we have
$$\mu \frac{s}{u} = \frac{f - r}{f' - r}. \tag{303}$$

From the triangles we have

$$\left.\begin{aligned} s^2 &= (f - r)^2 + r^2 + 2r\,(f - r) \cos \theta \\ &= f^2 - 2r\,(f - r) \text{ versin } \theta, \\ u^2 &= (f' - r)^2 + r^2 + 2r\,(f' - r) \cos \theta \\ &= f'^2 - 2r\,(f' - r) \text{ versin } \theta. \end{aligned}\right\} \tag{304}$$

Substituting u and s from these equations in (303), and reducing, we have

$$\left.\begin{aligned} &(f - r)\sqrt{f'^2 - 2r\,(f' - r) \text{ versin } \theta} \\ &\qquad = \mu\,(f' - r)\sqrt{f^2 - 2r\,(f - r) \text{ versin } \theta}. \end{aligned}\right\} \tag{305}$$

The relation existing between f and f' by this equation shows

that for every constant value of f the values of f' will vary with θ, and therefore rays proceeding from a radiant on the axis will not in general meet the axis, after deviation, in a single point; and for particular values of θ, every value of f will give a real value for f'.

The two distances f and f', being thus related, are called *conjugate focal distances.*

269. The surface of accurate refraction, which is independent of θ for a given radiant O, on the axis, is formed by revolving about the axis OA a curve such that, for any incident ray OI, OI′, etc., we have $OI + \mu IF = OI' + \mu I'F =$ a constant. Among the particular cases, we have

1°. When, if the incident rays are parallel, then the incident surface must be convex and a portion of a prolate spheroid, whose eccentricity is $e = \frac{1}{\mu}$.

2°. When the incident rays diverge from the further focus of an hyperboloid of revolution and the eccentricity of the generating curve is $e = \mu$, the deviated rays will then be parallel.

3°. The refraction at a spherical surface is accurate, for a diverging pencil at a concave surface, and for a converging pencil at a convex surface, when the radius is a mean proportional between the conjugate focal distances, estimated from the centre of curvature.

270. If versin θ be zero, we will have

$$\left.\begin{aligned}(f - r)f' &= \mu(f' - r)f,\\ \text{or,}\qquad f' &= \frac{\mu r f}{(\mu - 1)f + r};\end{aligned}\right\} \tag{306}$$

whence,

$$\frac{1}{f'} = \frac{\mu - 1}{\mu r} + \frac{1}{\mu f}, \tag{307}$$

an equation from which the distance of the geometrical focus f', corresponding to any radiant f, can be found.

For all pencils for which θ is so small as to make its versed-sine approximately zero, the above equation can be employed to find the conjugate focal distances.

Such pencils are called *small direct pencils.*

271. ***Application of Eq. (305) to Reflection at Plane Surfaces.*** Let BAD be any plane reflecting surface; then, in Eq. (305), θ becomes zero for all positions of the radiant O on the axis O′AO, $\mu = -1$, and Eq. (305) reduces to

$$f = -f'; \tag{308}$$

therefore the conjugate focus is as far behind the reflector as the radiant is in front, and the vergency of rays is not affected by reflection from plane surfaces. The image of any object seen by reflection appears to occupy a corresponding position behind the reflector, no matter where the eye be placed to view it; the only change being that the apparent right-hand side of the image corresponds to the left-hand side of the object.

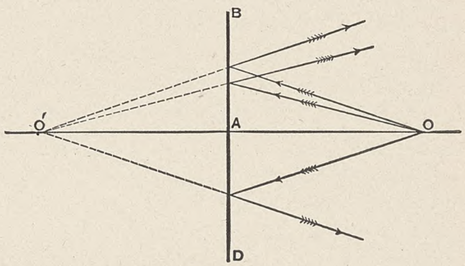

Figure 42.

Hence the image of a printed or written word presents the letters in an inverse order, from left to right, with the letters also reversed; the clear impression of a word on a blotter, when reflected by a plane mirror will, however, exhibit the letters and word in their proper form and arrangement. It is also evident that a second reflection of an object would reverse the first image and thus make the second image similar in all respects to the appearance of the object.

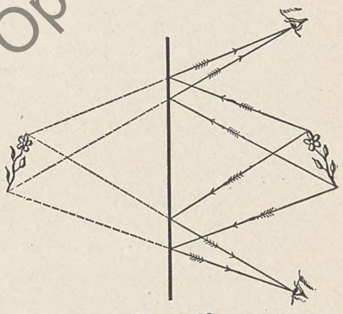
Figure 43.

Figure 43 shows the pencil of rays apparently proceeding from the virtual image formed by a plane reflector for two positions of the eye, and it is evident that wherever the eye be located in front of the reflector, it will receive a reflected pencil of the same vergency as if it were viewing a real object located at the position of the virtual image.

272. ***Multiple Reflection.*** When a radiant *P*, Figure 44, is seen by reflection in two parallel mirrors, the distance of the image P' from $P_{,}$, each being due to a single reflection, is double the distance separating the mirrors. Each image, P' and $P_{,}$, considered as a new radiant, will determine a new virtual focus at a distance behind the mirror which it faces equal to its distance in front, and so on; therefore the images of the point P will be theoretically unlimited in number, and will all be located on the normal to the mirrors through P. The loss of light by reflection will diminish the intensity of each successive image, and ultimately cause the luminosity of the succeeding ones to be inappreciable.

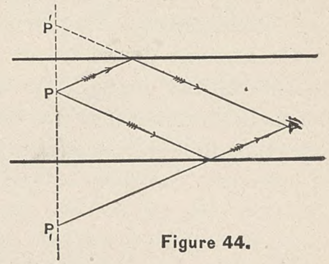

Figure 44.

273. ***Multiple Reflection by Inclined Mirrors.*** Let OM and OM′, Figure 45, be two plane mirrors inclined to each other at the angle *i*, and P be any luminous point in the included angle. With O as a centre, describe the circumference MPM′; then the two series of images of P formed by the mirrors M and M′ will be P', P'', P''', etc., and $P_{,}$, $P_{,,}$, $P_{,,,}$, etc. Regarding P as an image, the number of images formed will depend on the angle of the mirrors, the number increasing as the angle *i* diminishes. If the angle *i* be an aliquot part of 180°, the number is limited wherever P be located in the arc MPM′. If *i* be 60°, 90°, or 120°, the number of images will be six, four, and three, respectively. When MM′ is contained an entire number of times *n* in the circumference, there will be $n + 1$ images of P, counting P as one image, except when *n is even*, or when *n* is uneven and P is at the middle of MM′. In the latter exception, two images coincide and are considered as but one. The kaleidoscope is based on this principle of multiple reflection.

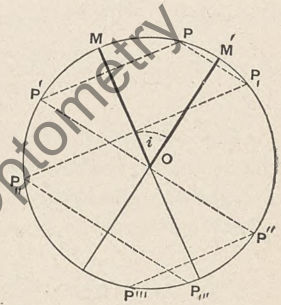

Figure 45.

274. ***Angular Velocity of the Reflected Ray.*** When a ray is incident on a plane mirror, and the latter is turned about an axis in its plane, the angular displacement of the ray is double that of the mirror; for the incident ray being constant in direction if the angular displacement of the mirror is α, the normal to the mirror has also turned through the angle α, and the angle of incidence has been either increased or diminished by the same angle; and since the angles of reflection and incidence are equal, the angle of reflection has been increased or diminished by the same value; therefore the angle that the reflected ray makes with the incident ray has been augmented or diminished by 2α, due to the change of direction of the mirror α.

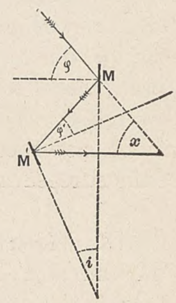

Figure 46.

275. When a ray is deviated by reflection by two mirrors inclined at an angle i to each other, the angle included between the original direction and the direction after the second reflection is double the angle included between the mirrors. Thus, in Figure 46, let ϕ and ϕ' be the first and second angles of incidence and x the required angle; then we have

$$\left.\begin{array}{l} 2\phi = 2\phi' + x, \qquad \text{and} \qquad \phi - \phi' = i; \\ \therefore \quad x = 2\phi - 2\phi' = 2i. \end{array}\right\} \qquad (309)$$

The sextant is constructed on this principle; M is the index-glass, and M′ the horizon-glass, and the angle x is read off on the circular arc, the units of which, for this purpose, are marked double their true value.

276. ***Deviation of Small Direct Pencils by Spherical Surfaces.*** Resuming Eq. (307), and limiting the discussion to small direct pencils deviated by any number of spherical deviating surfaces, we have

$$\frac{1}{f'} = \frac{\mu - 1}{\mu r} + \frac{1}{\mu f}.$$

If now we regard f' as a new radiant, its conjugate focus f'', after a second refraction by another medium bounded by a spherical surface, will be given by

$$\frac{1}{f''} = \frac{\mu' - 1}{\mu' r'} + \frac{1}{\mu' (f' + d)}, \tag{310}$$

in which f', f'', are the first and second focal distances from the origin, r' the radius of curvature of the second spherical surface, d the distance of the second surface from the first, measured on the axis of the pencil, and μ' the relative index of refraction of the second medium. In the same way, for any number of media, we will have for the focal distance,

$$\frac{1}{f^{n+1}} = \frac{\mu^n - 1}{\mu^n r^n} + \frac{1}{\mu^n (f^n + d^{n-1})}. \tag{311}$$

By the substitution of these values in succession, the final focal distance can be found. In ordinary cases, the thickness d can be neglected in comparison with the radiant distance f, and hence the equation can be written

$$\frac{1}{f^{n+1}} = \frac{\mu^n - 1}{\mu^n r^n} + \frac{1}{\mu^n f^n}. \tag{312}$$

If the pencil emerge from the first refracting medium into the incident medium, μ' in Eq. (310) will be equal to $\frac{1}{\mu}$, and we will have

$$\left.\begin{aligned} \frac{1}{f''} &= \frac{\frac{1}{\mu} - 1}{\frac{r'}{\mu}} + \mu\left(\frac{\mu - 1}{\mu r} + \frac{1}{\mu f}\right) \\ &= (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f}. \end{aligned}\right\} \tag{313}$$

Considering this focus f'' a new radiant, its conjugate focus, f^{iv}, after deviation by another medium, bounded by two other spherical surfaces, whose radii are r'' and r''', will be given by

$$\frac{1}{f^{\text{iv}}} = (\mu'' - 1)\left(\frac{1}{r''} - \frac{1}{r'''}\right) + (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f}, \tag{314}$$

and so on.

277. Representing principal focal distances by F, properly accented, we will have, since $f = \infty$ and $\frac{1}{f} = 0$,

$$\frac{1}{F_1} = \frac{\mu - 1}{\mu r}, \tag{315}$$

$$\frac{1}{F_2} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right), \tag{316}$$

$$\left.\begin{aligned} \frac{1}{F_4} &= (\mu'' - 1)\left(\frac{1}{r''} - \frac{1}{r'''}\right) + (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) \\ &= \frac{1}{F_2'} + \frac{1}{F_2}; \end{aligned}\right\} \tag{317}$$

and, generally,

$$\frac{1}{F_{2n}} = \frac{1}{F_2} + \frac{1}{F_2'} + \frac{1}{F_2''} + \text{etc.} = \Sigma \frac{1}{F_2}. \tag{318}$$

Substituting $\frac{1}{F_2}$ in Eq. (313) for its value, Eq. (316), we have

$$\frac{1}{f''} = \frac{1}{F_2} + \frac{1}{f}, \tag{319}$$

an equation suitable for the discussion of the deviation of a small direct pencil of light, by refraction by a medium whose bounding surfaces are spherical, and whose thickness can be neglected. This equation being of the first degree, and containing two variables, f and f'', there are an infinite number of sets of values of the conjugate focal distances which will satisfy it. It is usual to consider the radiant distance f as the independent variable, and the focal distance f'' as the function.

278. A *lens* is any medium bounded by *curved* surfaces, used to deviate light by refraction. The lenses ordinarily used have spherical surfaces, because of the readiness with which these surfaces can be ground and polished. Spherical lenses are either *concave* or *convex*.

The convex lenses are three in number, viz., the *double convex*, the *plano-convex*, and *concavo-convex* or *meniscus*, and are designated

in Figure 47 as 1, 2, and 3, respectively. There are also three concave lenses, viz., the *double concave*, the *plano-concave*, and the *convexo-concave*, designated as 4, 5, and 6 in the figure.

Figure 47.

279. The principal focal distance of a spherical lens being given by

$$\frac{1}{F_2} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right),$$

we see that it is constant for the same medium and the same curvature. Taking $\mu > 1$, as is the case when light enters from air into the several media of which the usual lenses are composed, and remembering the rule with respect to the sign of measured distances, we see that, from any origin,

$$\frac{1}{F_2} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right)$$

is negative for all convex, and positive for all concave spherical lenses. The numerical value of F_2 will, however, depend on the particular values of μ, r, and r'.

280. ***Discussion of the Properties of a Lens.*** By this is meant the determination of the focus corresponding to any position of the radiant taken on the axis of the lens.

1°. Convex lens. The equation is

$$\frac{1}{f''} = -\frac{1}{F_2} + \frac{1}{f}. \qquad (320)$$

By assuming values of f equal to $+\infty$, $+2F_2$, F_2, 0, and $-\infty$, we have, from the solution of Eq. (320), for the corresponding values of f'',

$$-F_2, \quad -2F_2, \quad -\infty, \quad 0, \quad \text{and} \quad -F_2;$$

and since the conjugate focal distances are connected by the law of continuity through Eq. (320), we see that as the radiant moves along the axis

from $+\infty$ to $+2F_2$,	the focus will move from	$-F_2$ to $-2F_2$;
" $+2F_2$ to F_2,	" " "	$-2F_2$ to $-\infty$;
" $+F_2$ to the lens,	" " "	$+\infty$ to the lens;
" the lens to $-\infty$,	" " "	the lens to $-F_2$.

Again, since positive values of f correspond to real radiants and negative values to virtual radiants, and negative values of f'' to real foci and positive values to virtual foci, the above results may be grouped in the diagram of Figure 48.

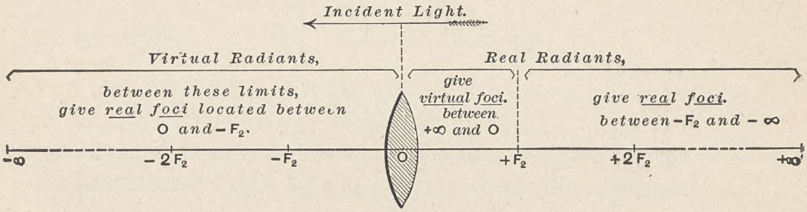

Figure 48.

2°. Concave lens. Its equation is

$$\frac{1}{f''} = +\frac{1}{F_2} + \frac{1}{f}. \tag{321}$$

The special values of the radiant distances necessary to be considered are

$$+\infty, \quad 0, \quad -F_2, \quad -2F_2, \quad -\infty,$$

for which the corresponding focal distances are

$$+F_2, \quad 0, \quad -\infty, \quad +2F_2, \quad +F_2;$$

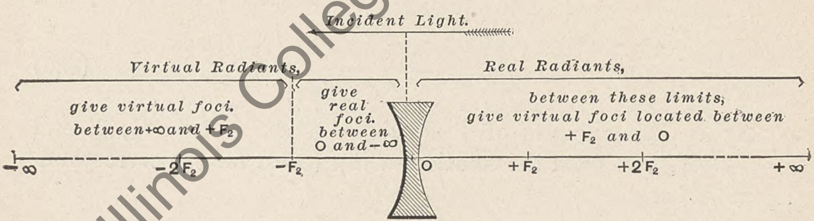

Figure 49.

and in the diagram of Figure 49 for this lens, it will be noticed that there is a complete analogy, if the words *virtual* and *real* be interchanged in the convex and concave lenses.

281. *Relative Velocities of the Radiant and Focus.* Differentiating the equation for lenses,

$$\frac{1}{f''} = \pm \frac{1}{F_2} + \frac{1}{f},$$

and dividing both members by dt, we have

$$\frac{df''}{dt} \cdot \frac{1}{f''^2} = \frac{df}{dt} \cdot \frac{1}{f^2}; \tag{322}$$

or replacing the differential coefficients by their equals V'' and V, we have

$$\frac{V''}{f''^2} = \frac{V}{f^2}. \tag{323}$$

Since the ratio of the velocities is positive, whatever be the values of f and f'', we see that the motion of the conjugate foci is in the same direction, and that their velocities are directly proportional to the squares of their distances from the lens.

282. *Discussion of Spherical Reflectors.* Eq. (307) gives the circumstances of the deviation of light at a single surface, and which in case of reflection becomes, since $\mu = -1$,

$$\frac{1}{f'} = \frac{2}{r} - \frac{1}{f}; \tag{324}$$

and since r is positive, the reflector is concave. Making $f = \infty$, we have, for the principal focal distance, $F_1 = \frac{r}{2}$; therefore, the equation for a concave reflector is

$$\frac{1}{f'} = \frac{1}{F_1} - \frac{1}{f}. \tag{325}$$

For a convex reflector, r being negative, the equation takes the form of

$$\frac{1}{f'} = -\frac{2}{r} - \frac{1}{f}, \tag{326}$$

or
$$\frac{1}{f'} = -\frac{1}{F_1} - \frac{1}{f}. \tag{327}$$

In reflectors, when the origin is taken at the reflector, it is easily seen that real radiants and real foci have the positive sign, and virtual radiants and virtual foci have the negative sign.

Making the discussion as in lenses, we readily obtain the following conclusions, viz.:

Concave Reflector. Beginning with a real radiant at $+\infty$, the focus will be real and at F_1; as the radiant moves toward the reflector, the focus moves in the opposite direction, these points passing each other at the centre of curvature; real radiants from this point to $+F_1$ correspond to real foci, located between the centre and $+\infty$; as the radiant approaches the reflector, the focus changes from real to virtual, and approaches the reflector from $-\infty$, both radiant and focus meeting at the reflector; finally, the radiant being virtual and proceeding to $-\infty$, gives real foci between the reflector and $+F_1$.

These relations are shown in the diagram of Figure 50.

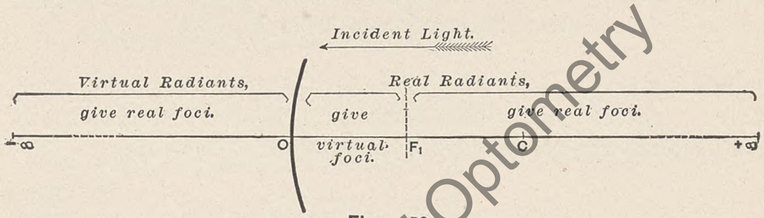

Figure 50.

Convex Reflector. A similar discussion of the equation

$$\frac{1}{f'} = -\frac{1}{F_1} - \frac{1}{f},$$

will show that all real radiants from $+\infty$ to the reflector give virtual foci, located between $-F_1$ and the reflector; that all virtual radiants between the reflector and $-F_1$ give real foci, located between the reflector and $+\infty$; and that all virtual radiants between $-F_1$ and $-\infty$ give virtual foci between $-\infty$ and $-F_1$. These results are grouped in Figure 51.

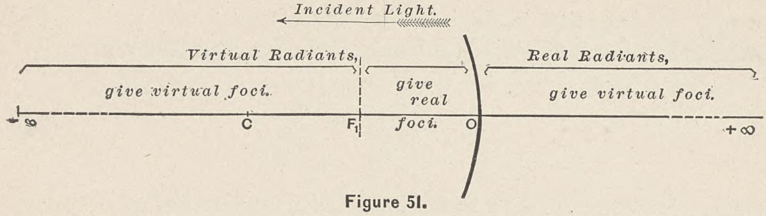

Figure 51.

283. Differentiating the equation for reflectors,

$$\frac{1}{f'} = \pm \frac{1}{F_1} - \frac{1}{f}, \tag{328}$$

dividing by dt, and replacing the differential coefficients by the velocities, we have

$$\frac{V'}{f'^2} = -\frac{V}{f^2}; \tag{329}$$

from which we see that the velocities of the conjugate foci are directly proportional to the squares of their distances from the reflector; and that the motion of these points is in opposite directions.

284. From the preceding discussions it is evident that the convex lens and concave reflector increase the convergency of incident light, and therefore either change incident diverging pencils into converging ones, or make them less diverging; also, that the concave lens and convex reflector increase the divergency and diminish the convergency of incident light. Therefore, the office of the convex lens and concave reflector is to *collect* rays, and that of the concave lens and convex reflector to *separate* them.

285. ***Power of a Lens.*** The expressions

$$\frac{1}{f} \text{ and } \frac{1}{f''}$$

may be taken to measure the vergency of the incident and deviated pencils, respectively; for, since the pencils are small and direct, f and f'' may be considered as the radii of the same arc, subtended by the incident and refracted pencils at the distance of the lens from these points; hence their reciprocals will have the same ratio as the angles of the pencils themselves. By the *power* of a lens is

meant its capacity to deviate that ray of the small pencil which meets the lens at the unit's distance from the axis, and since

$$\frac{1}{f''} - \frac{1}{f} = \pm \frac{1}{F_2}, \tag{330}$$

we see that the vergency after, diminished by that before deviation, and which is evidently the vergency increment, is for the same lens constant and equal to the reciprocal of the principal focal distance. Hence, to ascertain the power of a lens, we have the following rule: Draw that ray of the pencil which meets the lens at the unit's distance from the axis, and find the corresponding conjugate focus of the radiant; the angle between the incident ray prolonged and the refracted ray is the power of the lens. Or, find the principal focal distance of the lens, and take its reciprocal. Since the deviation of each ray of the pencil varies with its inclination to the axis, it is better to find the deviation of the ray at a unit's distance, although any other ray would answer, not too near the axis; thus the axial ray is undeviated, the next is but slightly deviated, and so on to the extreme ray, which is deviated the most.

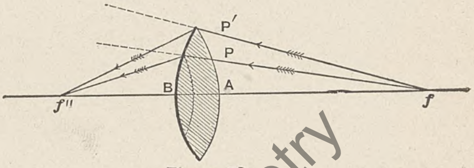

Figure 52.

It is also evident that two lenses, constructed of the same material and with faces having the same curvature, are of the same power. Thus, the two lenses, A and B, Figure 52, will bring the divergent pencils P and P′ to the same focus f''.

286. ***To find the Principal Focal Distance of a Lens.*** Since we have

$$\frac{1}{F_2} = \pm (\mu - 1)\left(\frac{1}{r} - \frac{1}{r'}\right), \tag{331}$$

it is evident that the power of the lens depends upon the refractive index and the curvature of its faces. When these are known, the power, and hence the principal focal distance, can be readily found.

Thus, the value of $\frac{1}{F_2}$ for a meniscus crown glass lens whose radii are 6 and 3 feet, respectively, and refractive index 1.525, is

$$.525\left(\frac{1}{6}-\frac{1}{3}\right)=-\frac{1}{11.42};$$

therefore the principal focal distance is negative, and 11.42+ ft. from the lens. Practically the principal focal distance of a negative lens can be found by measuring the distance of the focus of a beam of sunlight from the lens, since these rays may be taken as parallel before deviation, and the radiant as at an infinite distance; or, since when the radiant is at a distance of $+2F_2$ the conjugate focus is formed at a distance of $-2F_2$, these positions can be found experimentally, and one-fourth of their distance apart will be the value of F_2. The principal focal distance of positive lenses can be found by associating them with negative lenses of greater power, finding the principal focal distance of the combination by either of the above methods, and subtracting from the power of the combination the power of the negative lens; the remainder will be the power of the positive lens, and its reciprocal the principal focal distance. This method depends on the equation

$$\frac{1}{F_{2n}}=\Sigma\frac{1}{F_2}. \tag{332}$$

287. For reflectors, we have

$$\frac{1}{f'}+\frac{1}{f}=\pm\frac{1}{F_1};$$

whence, the sum of the vergencies before and after deviation is equal to the power of the reflector. The principal focal distance of a spherical reflector being equal to half its radius of curvature, there is no difficulty in finding its power. When the reflector is plane, $\frac{1}{F_1}$ becomes zero, and we have, as in Art. 271,

$$f'=-f.$$

Hence, plane reflectors have no power to change the vergency of the incident pencil.

288. ***Deviation of Oblique Pencils by Reflection or Refraction at Spherical Surfaces.*** When the axis of the pencil is not coincident with the axis of the deviating surface, the pencil is said to be *oblique.* When the axis of the oblique pencil meets the surface at a point with reference to which the area covered by the pencil is symmetrical, the incidence is called *centrical,* and when otherwise, *excentrical.* We have seen, Art. 268, that the rays from a single radiant on the axis of the deviating surface are not in general brought to a single focus, but that by Eq. (305) we can find the particular direction of any deviated ray. The principles enunciated in Art. 85 show that the consecutive deviated rays will be tangent to some surface, which is the locus of their consecutive intersections, and which may therefore be regarded as their envelope. This surface is called the *caustic surface.* The general problem of caustic surfaces, as outlined in Art. 85, is one of considerable intricacy, and it is sufficient to consider only those of deviating surfaces of revolution, since such surfaces are usually employed in optical instruments. A surface of revolution has in general two generatrices; one being the meridian curve, and the other a circle in a normal plane to the meridian curve.

If we consider any indefinitely small area of the deviating surface, we see that the deviated rays will form two systems of developable surfaces at right angles to each other, when the surface is conceived to be generated by the curves above referred to. Each deviated ray will belong at the same time to both systems, and the consecutive intersections of the rays in a plane of each system will form a curve of regression, which will be the caustic curve cut out of the caustic surface by the plane in question. To illustrate, let HAK, Figure 53, be a portion of a spherical deviating surface, whose axis CQ contains the radiant Q. We may regard the incident pencil as being made up of a series of conical surfaces of rays, whose angular spread θ on the surface varies from the extreme value HOC by continuity to zero. Each cone of rays will have a single focus, such as r, q_2, s, since, by Eq. (305), f' is a function of (f, μ, θ) only, and the consecutive foci of these cones will be spread over the axis from that point corresponding to the extreme cone to the geometrical focus of the small direct pencil. The intersections of the consecutive deviated cones will form consecutive curves of the caustic surface, whose planes are normal to the axis of the de-

viating surface. Consider now the annular area of the deviating surface formed by revolving the indefinitely small area HAK around the axis CQ. The corresponding annulus of the caustic surface will be at the limit the arc of a circle, and when HK is very small it may be taken as such, the diameter of which in the figure is q_1t. If we consider the small beam of light incident on the area HAK alone, it may be taken as a small oblique pencil centrically incident, whose axis after deviation will be Aq_1q_2. Every deviated ray will

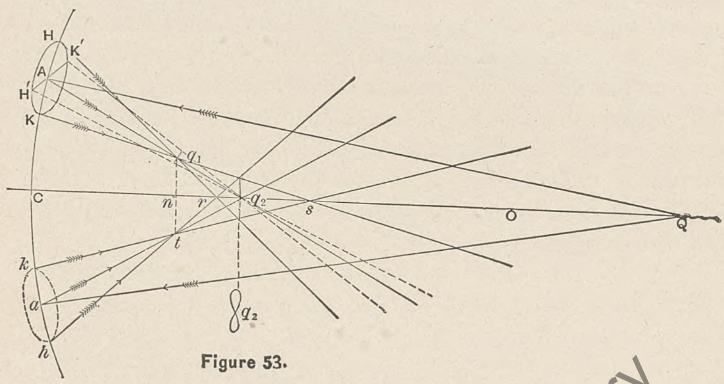

Figure 53.

now pass through a point of the small circular arc at q_1, which may be regarded as coincident with its tangent and normal to the plane QCA. This line and the point q_1 are called the *primary focal line* and the *primary focus*, because the points of the line are the *first* intersections of the consecutive deviated rays. Again, considering the second generatrix H'AK' of the area, every deviated ray will pass through the corresponding part of the intersection of the caustic surface made by the plane $H'K'q_2$ at the point q_2, and this intersection and point q_2 are called the *secondary focal line* and *secondary focus* of the oblique pencil. The secondary focus is then the intersection of the axis of the oblique pencil and the axis of the deviating surface. The secondary focal line, although like an elongated figure 8, may be taken approximately as a right line. The plane containing the axis of the incident pencil and the axis of the deviated pencil, or the plane QCA, is called the *primary plane;* and the plane containing Aq_1q_2, normal to the primary plane, is the *secondary* plane. Hence, we may conclude that a small oblique

pencil, after deviation, converges to or diverges from two *right lines* called *focal lines*, the primary and secondary focal lines being normal to the primary and secondary planes, respectively. When the deviation is by reflection at a single surface the focal lines are real, and by refraction, when $\mu > 1$, virtual. That property of a deviating surface by which the rays of a pencil are caused to pass through the two focal lines after deviation is called *astigmatism*.

289. ***Circle of Least Confusion.*** If the deviated pencil be cut by a plane parallel to the tangent plane to the surface at A, we see that at q_1 the section will be a right line, normal to the primary plane; and as the cutting plane is moved towards q_2, the breadth of the section increases in the primary plane and decreases in the secondary, until at q_2 the section becomes a right line in the primary plane. The section, which at some point between q_1 and q_2 is nearly circular, is called the *circle of least confusion*, and may be taken as the approximate focus of the oblique pencil.

290. ***The Positions of the Foci.*** In Figure 54, let Q be the radiant from which the small oblique pencil, whose axis is QA, is incident centrically upon the refracting surface at A; O the centre of the surface, and μ the relative index. The intersection q_2 of the ray through O with QA, after deviation, is the secondary focus; q_1, where any ray of the primary plane, after deviation, intersects the axis Aq_2 of the deviated pencil, is the primary focus.

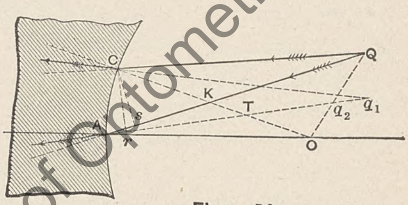

Figure 54.

Draw Cs and Cr perpendicular to AQ and Aq_2, respectively, and let K and T indicate the angles made by the intersections of OC and AQ, and OC and Aq_2, respectively; let AQ $= f$, $Aq_1 = f_1$, $Aq_2 = f_2$, AO $= r$, QOA $= \theta$, ϕ and $\phi + d\phi$ the angles of incidence, and ϕ' and $\phi' + d\phi'$ the angles of refraction of QA and QC, respectively. Then we have

$$\text{CQA} = \frac{\text{C}s}{\text{CQ}} \text{ nearly} = \frac{\text{AC} \cos \phi}{f}, \tag{333}$$

$$Cq_1A = \frac{AC \cos \phi'}{f_1}, \tag{334}$$

$$\left.\begin{aligned} d\phi &= QCO - QAO \\ &= \text{angle K} - CQA - (\text{angle K} - AOC) \\ &= AOC - CQA = \frac{AC}{r} - \frac{AC \cos \phi}{f}, \end{aligned}\right\} \tag{335}$$

$$\left.\begin{aligned} d\phi' &= q_1CO - q_1AO \\ &= \text{angle T} - Cq_1A - (\text{angle T} - AOC) \\ &= AOC - Cq_1A = \frac{AC}{r} - \frac{AC \cos \phi'}{f_1}. \end{aligned}\right\} \tag{336}$$

Differentiating $\sin \phi = \mu \sin \phi'$, we have

$$\cos \phi \, d\phi = \mu \cos \phi' \, d\phi'. \tag{337}$$

Substituting the values of $d\phi$ and $d\phi'$, we have

$$\left(\frac{1}{r} - \frac{\cos \phi}{f}\right) \cos \phi = \left(\frac{1}{r} - \frac{\cos \phi'}{f_1}\right) \mu \cos \phi', \tag{338}$$

$$\frac{\mu \cos^2 \phi'}{f_1} - \frac{\cos^2 \phi}{f} = \frac{\mu \cos \phi' - \cos \phi}{r}, \tag{339}$$

from which the distance of the primary focus f_1 from A can be found.

For the secondary focus we have

$$\frac{r}{f_2} = \frac{AO}{Aq_2} = \frac{\sin (\phi' + \theta)}{\sin \theta} = \cos \phi' + \sin \phi' \cot \theta \tag{340}$$

$$\frac{r}{f} = \frac{AO}{AQ} = \frac{\sin (\phi + \theta)}{\sin \theta} = \cos \phi + \sin \phi \cot \theta, \tag{341}$$

Eliminating $\cot \theta$, and reducing by $\sin \phi = \mu \sin \phi'$, we have

$$\frac{\mu}{f_2} - \frac{1}{f} = \frac{\mu \cos \phi' - \cos \phi}{r}. \tag{342}$$

For refraction at plane surfaces, $r = \infty$, and the second members of Equations (339) and (342) reduce to zero. For reflection, $\mu = -1$, and these equations become

$$\frac{1}{f_1} + \frac{1}{f} = \frac{2}{r \cos \phi}, \qquad (343)$$

$$\frac{1}{f_2} + \frac{1}{f} = \frac{2 \cos \phi}{r}. \qquad (344)$$

291. The position of the circle of least confusion and the value of its diameter can readily be found from the properties of similar triangles; the resulting values will be given in terms of the length and breadth of the beam on the deviating surface, as determined by the intersections of the primary and secondary planes, and in terms of the focal distances measured on the axis of the deviated pencil from the surface. The limiting position for the centre of the circle is midway between f_1 and f_2, at which point it may be assumed for all pencils of small obliquity.

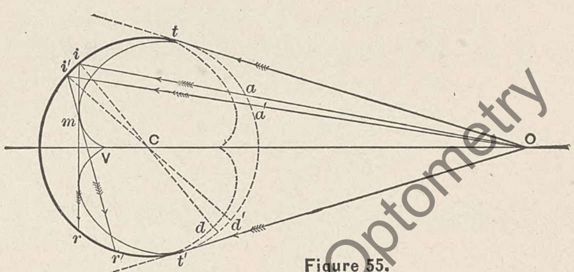

Figure 55.

292. ***Particular Cases of Caustic Curves.*** Let trt', Figure 55, be a concave spherical reflector, and O any radiant on its axis; Oi and Oi' any two consecutive rays, and m their point of intersection after reflection; then, from Eq. (343), we have at once, by representing ia by $4c$,

$$\frac{1}{f_1} + \frac{1}{f} = \frac{1}{c}; \qquad (345)$$

and if the reflector be convex, f_1 and c are negative, and we have

$$\frac{1}{f} - \frac{1}{f_1} = -\frac{1}{c}. \qquad (346)$$

The caustic curves may, from any assumed position of the radiant, be constructed by points, and when revolved about the axis they will generate the caustic surface for the spherical reflector. For parallel rays, f is ∞ and $f_1 = c$. In this case, Figure 56, the curve is an epicycloid generated by a point whose initial position is v, in the circumference of the small circle Av, with a radius equal to $\frac{r}{4}$, rolling upon the circumference vp, whose radius is $\frac{r}{2}$, r being the radius of the spherical surface. To show this, let m be any point of the caustic; then

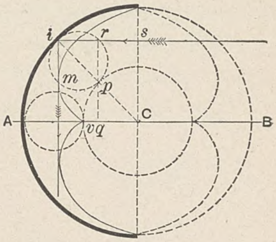

Figure 56.

$$im = \frac{r}{2}\cos\phi = f_1.$$

Again, the arc pm is equal to the arc pv, since the angles pim and pcv are equal. The virtual caustic arising from reflection on the convex surface indicated by the dotted curve is symmetrical with and equal to the real caustic. As the radiant moves from ∞ along the axis, the cusp v approaches the centre C, the virtual caustic diminishes until the radiant reaches B, when it is zero, and the real caustic springs from B in both directions to v. In this case, the radius of both circles is $\frac{r}{3}$, as shown in Figure 57; for we have

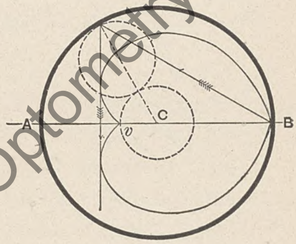

Figure 57.

$$\frac{1}{f} = \frac{1}{2r\cos\phi}$$

when the radiant is at B, which substituted in Eq. (343) gives

$$\left.\begin{aligned} \frac{1}{f_1} &= \frac{2}{r\cos\phi} - \frac{1}{2r\cos\phi} = \frac{3}{2r\cos\phi}; \\ \therefore\ f_1 &= \frac{2}{3}r\cos\phi. \end{aligned}\right\} \qquad (347)$$

As the radiant moves within the spherical surface, the cusp approaches the centre, uniting with the radiant at this point, and the caustic reduces to zero. For radiants on the axis within the surface, the caustic has a point of regression on each side of the axis near the reflector, from which asymptotic branches proceed, and a virtual caustic of two branches is formed exterior to the reflector on the convex side. Figures 58, 59, and 60 represent these

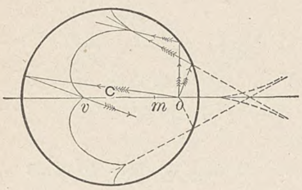

Figure 58.

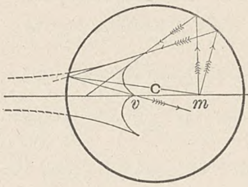

Figure 59.

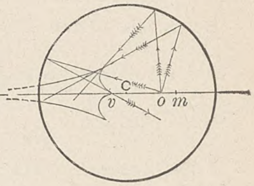

Figure 60.

caustics for positions of the radiant within the spherical surface at distances from the centre $> \frac{r}{2}$, $= \frac{r}{2}$, and $< \frac{r}{2}$, respectively. All these, as well as other special cases, can be obtained from the general principles enunciated in Art. 86.

293. ***Critical Angle.*** From the equation

$$\sin \phi = \mu \sin \phi',$$

since μ is greater than unity when the incident ray is in air, $\sin \phi'$ is always less than unity for all angles of incidence, and therefore $\phi' < 90°$. Then at all angles of incidence, when light passes from a rarer to a denser medium, there will be refraction; but when the light passes from a denser to a rarer medium, there can be no refraction when the angle of incidence surpasses the value

$$\phi' = \sin^{-1} \frac{1}{\mu}.$$

Rays making greater angles of incidence than this are totally reflected. This angle is called the *limiting* or *critical* angle of incidence for refraction. Its value for water is

$$\sin^{-1} \frac{1}{1.336} = 48° \, 27' \, 40'';$$

for crown glass about 40° 30′, and for chromate of lead about 19° 28′ 20″. Therefore, to an eye placed in water, Figure 61, all external objects would appear crowded within a conical space NA, of about 48° 28′ on either side of the normal to the surface.

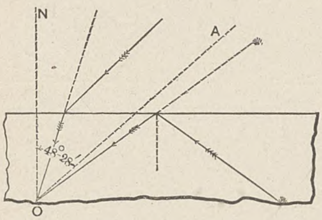

Figure 61

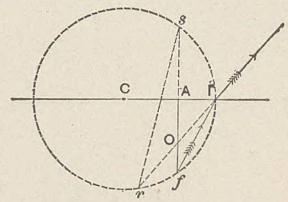

Figure 62.

The remaining angle of 41° 32′ would be filled with rays, proceeding from objects in the water, which have been totally reflected at the surface.

Let us now consider rays refracted from a denser into a rarer medium. Let f, Figure 62, be the radiant, in water, for example, and consider any ray whose angle of incidence is less than 48° 28′, incident at I. Draw fAs normal to the surface, and make As = Af. Describe a circle through f, I, and s, and construct the lines as in the figure. We have, geometrically,

$$\frac{\mathrm{O}s}{\mathrm{O}f} = \frac{rs}{rf},$$

and

$$\therefore \quad \frac{sf}{\mathrm{O}f} = \frac{rs + rf}{rf};$$

$$\therefore \quad \frac{sf}{rs + rf} = \frac{\mathrm{O}f}{rf} = \frac{\sin fr\mathrm{I}}{\sin f\mathrm{O}r} = \frac{\sin \phi}{\sin \phi'} = \mu;$$

$$\therefore \quad rs + rf = \frac{sf}{\mu} = \text{a constant.}$$

The locus of r is therefore an ellipse, whose foci are s and f; rI is normal to the ellipse at every point r, since it bisects the angle included between the foci.

The caustic curve is therefore the evolute of the ellipse whose foci are s and f, and transverse axis $\frac{sf}{\mu}$.

By constructing the caustic for any radiant, as in Figure 63, we see that the rays emerging into a rarer medium are tangent to the caustic curve, and hence the radiant Q in the denser medium would appear to an eye in the rarer medium lifted up to Q′, nearer to the separating surfaces, and along the successive tangents to the caustic curve, depending on the position of the eye. If PQ, Fig. 64, be any object, as a straight rod, parallel to the surface of water, points of the image P′Q′ may be constructed as in the figure, showing that, to an eye in the air, the right line appears curved, and that the portion directly under the place of the eye is apparently deeper in the water than the more remote portions.

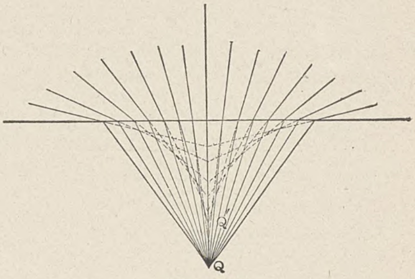

Figure 63

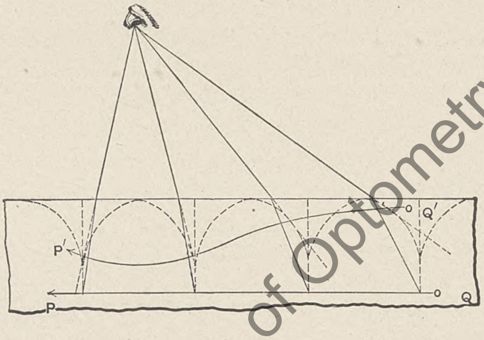

Figure 64.

The horizontal lower surface below clear water presents, for the same reason, a basin-shaped appearance.

294. ***Spherical Aberration.*** In Figure 65, let F, be the principal focus of the spherical reflector whose centre is C, and x the point in which the extreme ray cuts the axis after deviation; then, since the triangle CIx is isosceles, the perpendicular from x to CI will bisect the radius at s. The triangles TIA and xsF, are

similar, and hence $F_{,}x$ is equal to $\frac{1}{2}$TA, or approximately to $\frac{1}{2}$AN. But AN is nearly equal to $\frac{AI^2}{2AC}$, and therefore the distance of the focus x from the limiting position $F_{,}$ varies as the square of the semi-aperture directly and as twice the diameter of the surface inversely.

This distance is called the *longitudinal*, and $F_{,}z$ the *lateral spherical aberration.*

These departures of large pencils after deviation from a single focus F, is due to the form of the deviating surfaces alone, and hence is called *spherical* aberration.

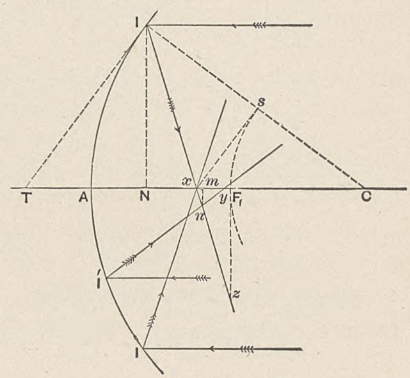

Figure 65.

As the incident ray approaches the axis, the focus approaches $F_{,}$, as at y. The point n, where that one of the intermediate rays which intersects the extreme ray at the furthest distance from the axis AC, limits the extent of the circular area through which all the deviated rays pass. This circle is called the *least circle of aberration.* To find the value of its radius, let α, α', represent the arcs AI and AI′, and a the aberration $F_{,}x$. Then, if the ray Ix be invariable and I′y vary, mn will vary directly with xm, and we have only to find when xm will be a maximum.

The aberration a and xy give

$$xy = a\frac{\alpha^2 - \alpha'^2}{\alpha^2}. \tag{348}$$

and we have, from the figure,

$$mn = xm\frac{\alpha}{Ax}, \tag{349}$$

$$my = mn\frac{Ay}{\alpha'} = xm\frac{Ay}{Ax}\cdot\frac{\alpha}{\alpha'}. \tag{350}$$

Taking $Ay = Ax$, we have approximately,

$$xy = xm + xm\frac{\alpha}{\alpha'} = \frac{\alpha' + \alpha}{\alpha'}xm. \tag{351}$$

Equating these values of xy, we have

$$\left.\begin{aligned} a\frac{\alpha^2 - \alpha'^2}{\alpha^2} &= \frac{\alpha' + \alpha}{\alpha'}xm, \\ \text{or} \qquad xm &= a\frac{\alpha'}{\alpha^2}(\alpha - \alpha'), \end{aligned}\right\} \tag{352}$$

which is a maximum for $\alpha' = \frac{\alpha}{2}$, and for this value we have

$$xm = \frac{a}{4} \quad \text{and} \quad mn = \frac{\mathrm{AI}}{\mathrm{A}x} \cdot \frac{\mathrm{F}_{,}x}{4} = \frac{\mathrm{F}_{,}z}{4}. \tag{353}$$

Hence, the least circle of aberration is at three-fourths of $\mathrm{F}_{,}x$ from $\mathrm{F}_{,}$, and its radius is one-fourth the lateral aberration.

The longitudinal aberration being $\frac{\alpha^2}{4R}$, from Eq. (353) we see that the lateral aberration will be approximately $\frac{\alpha^3}{2R^2}$.

295. In the discussion for lenses, we have omitted all consideration of the thickness, and limited the question to the deviation of small pencils, and hence the results are only approximately true. The more complete investigation shows that the aberration not only depends on the thickness of the lens, but upon the particular surface which is first presented to incident light in the same lens. On this account it is sometimes usual, in naming lenses, to designate the first incident surface; thus, *plano-concave* indicates that light is incident on the plane surface, while *concavo-plane* is the same lens with light incident on the concave surface. This designation has not been followed in the text, because our limits have not permitted more than a passing allusion to the subject of aberration. When in a single lens, or in a combination of lenses, the aberration is destroyed, the lens or combination is said to be *aplanatic*. In the former case, the lens is usually a meniscus of a particular form for a beam proceeding from a determinate point as a radiant; in the latter, the aberration of one lens counteracts or destroys the aberration of the others. The ocular of Huyghen's, referred to hereafter,

is an example of an approximate aplanatic combination. Surfaces such as the ellipsoid, paraboloid, and hyperboloid are of course free from spherical aberration, so that the rays of incident pencils, having their radiants at certain points, meet after deviation accurately in another point. From the geometrical properties of these surfaces, and the laws of refraction and reflection, the circumstances of the deviation of light can readily be determined. The difficulty of accurately forming these surfaces gives them only a theoretical interest, and the discussion of their properties is for this reason generally omitted. In grinding and polishing the surfaces of lenses and reflectors, it is not always possible for the practical optician to attain the best mathematical form of the surfaces, nor is it always desirable, since the lack of complete homogeneity assumed in theory is rarely attained in practice. A great part of the ultimate value of a lens depends on the skill of the optician in removing the defects which crop out in the course of preparation, and for which no merely theoretical discussion can provide.

296. ***Optical Centre.*** The point in which the line joining the extremities of two parallel radii intersects the axis of a lens is called its *optical* centre. Let OAA′, Figure 66, be the axis of a lens, O and O′ the centres of the first and second surfaces, I and I′ the extremities of any two parallel radii. Join I and I′, and let C be the point in which II′ meets the axis; this will be the optical centre. To find its distance from A, let r and r' be the radii of the surfaces, x the distance AC, and t the thickness of the lens. By similar triangles, we have

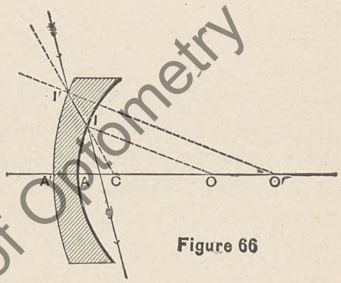

Figure 66

$$\frac{CO}{CO'} = \frac{r}{r'} \quad \text{or} \quad \frac{r - x}{r' - t - x} = \frac{r}{r'};$$

$$\therefore \quad x = \frac{rt}{r' - r}. \qquad (354)$$

A similar result can be obtained from Figure 67, and we see that the distance of the centre from the lens depends only on the radii

of its surfaces and the thickness of the lens, and that it is independent of the refractive index and therefore of the material of which the lens is constructed. By substituting the values of r, r', and t, in the value of x, and paying attention to their signs, we readily find:

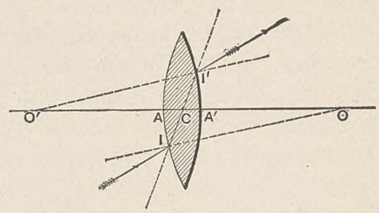

Figure 67

1°. That, if the curvatures of the surfaces are in opposite directions, the centre lies within the lens.

2°. If one surface be plane, the centre is on the curved surface.

3°. If the curvatures are in the same direction, the centre is outside of the lens, and on the side of the greater curvature.

4°. If the thickness t be negligible, the centre may be assumed at any point of the lens on the axis.

The optical centre of a spherical reflecting surface is by definition at the centre of curvature of the surface. The equation of the small pencil, when the focal distances are measured from this point as an origin, can be obtained readily from Eq. (307), and is

$$\frac{1}{f'} = \mp\frac{2}{r} - \frac{1}{f}, \tag{355}$$

for concave and convex reflectors.

When a ray is refracted through a lens in such a manner that its direction between the two refractions passes through the optical centre, the ultimate direction of the ray is parallel to its primitive direction; for it is in the same condition as if it had passed through a portion of the medium bounded by parallel planes; and when the lens is thin and the obliquity small, the lateral displacement is negligible. Therefore, if we draw a ray from any radiant through the optical centre of a lens or reflector, we may consider it as the axis of a small oblique pencil on which the other rays of the pencil met by the lens or reflector are brought to a focus. This property is made use of in constructing the images of objects formed by a lens or reflector.

297. ***Focal Centres of a Lens.*** The limiting positions on the axis of the lens, where the directions of the incident and

emergent ray cut the axis, when the direction of the ray while passing through the lens contains the optical centre, are called the *focal centres* of the lens. Supposing, in Figure 66, the incident ray through I′ be prolonged and its point of intersection with the axis be designated by m, and that the point where the emergent ray cuts the axis be designated by n; then passing to the limit, that is, when these rays coincide with the axis, Eq. (307) is applicable, and we have

$$\frac{\mu}{AC} - \frac{1}{Am} = \frac{\mu - 1}{r}, \qquad (356)$$

$$\frac{\mu}{A'C} - \frac{1}{A'n} = \frac{\mu - 1}{r'}; \qquad (357)$$

also,

$$AC = \frac{rt}{r' - r}, \qquad (358)$$

and by combination,

$$Am = \frac{rt}{\mu\,(r' - r - t) + t}, \qquad (359)$$

$$A'n = \frac{r't}{\mu\,(r' - r - t) + t}, \qquad (360)$$

hence, these distances depend on the refractive index, thickness and radii of curvature of the lens.

298. ***The Eye.*** The human eye, Figure 68, is very nearly spherical in form, the front part being slightly protuberant. The globe of the eye consists of the *sclerotic coat*, known generally as the *white of the eye*, which covers all but a small part of the anterior surface, and the *cornea*, a thin transparent membrane covering the front, through which the *iris* and *pupil* are seen. The iris gives color to the eye, and is a membranous plate crossing the axis of the eye transversely from the junction of the sclerotic coat and the cornea. The dark circular aperture in the iris or the pupil permits the passage of light, and is capable of involuntary adjustment, becoming larger or smaller as the intensity of light is little or great.

The interior of the eye behind the iris is lined with two thin membranes. That next the sclerotic coat is called the *choroid*, and has its cells filled with a dark pigment, giving to it a brownish

black velvety appearance. The iris is a part of the anterior portion of the choroid coat, drawn together like a curtain, and is thickened by muscular tissue. The other part of the choroid coat surrounds the outer edge of the crystalline lens in a series of folds, about seventy in number, and is called the *ciliary processes.*

The membrane lying within the choroid coat, and resting upon it, is called the *retina.* It is really a nerve tissue completely lining the interior of the eye, as far forward as the ciliary processes, where it forms an irregular border. Near the point where the axis of the eye meets the retina is the *yellow spot,* which has a diameter of about $\frac{1}{20}$ of an inch. One-sixth of an inch from this point towards the nose is the centre of the optic nerve; this portion of the retina where the optic nerve enters is called the *punctum cœcum,* not being sensitive to light.

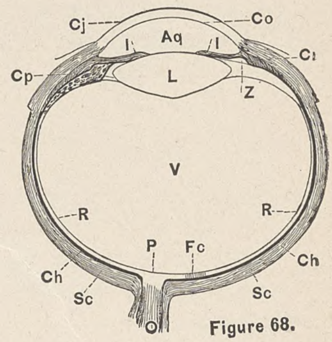

Figure 68.

SECTION OF THE EYE.

Cj, conjunctiva; *Co*, cornea; *Aq*, aqueous humor; I, iris; *Cm*, ciliary muscle; *Cp*, ciliary processes; Z, zonule of Zinn; L, crystalline lens; V, vitreous humor; R, retina; *Fc*, yellow spot; P, blind spot; *Ch*, choroid; *Sc*, sclerotic; O, optic nerve.

By making two dark spots on a piece of paper a few inches apart, and directing the axis of either eye to the nearer spot, the other eye being closed, the outer spot will disappear when the paper is at the proper distance from the eye; the rays from this spot then meet the retina at the punctum cœcum of the open eye. A displacement of the paper in either direction brings the spot again into view.

The retina consists of nine layers of nerve fibres, cells, tissues, and blood vessels, forming a very complex structure.

The layer nearest the centre of the eye, called the *baciliary layer,* consists of numerous elongated elements, arranged side by side, normal to the surface of the retina. Some of these elements are cylindrical, called rods, and others are conical and shorter than the rods. At the yellow spot there is a slight depression, and the rods are replaced by cones, closer in arrangement, longer but more slender in form than the other cones; this part of the eye is by far the most sensitive to light.

The humors of the eye are three in number. Their object is to refract the light, and bring it to a focus on the retina. The first in order, beginning at the front of the eye, is the *aqueous*, which is a watery fluid, having a little sodium chloride in solution. It occupies the space between the cornea and the front of the *crystalline lens*. The latter is contained in a membranous double-convex capsule, the greater curvature being to the rear; its thickness is about one-third its transverse diameter. The crystalline lens consists of several nearly concentric layers of varying density, decreasing from the centre outwards; the transparency of this lens is destroyed by the disease known as *cataract*. The *vitreous humor* fills the larger volume of the eye, from the crystalline lens in front to the retina; it is clearer than the most transparent glass, and its consistency is like that of jelly.

The eye, as an optical instrument, is similar in its function to the camera obscura. The sclerotic coat forms the wall of the chamber; the choroid coat, the dark lining for the absorption of surplus light; the cornea and humors, the system of lenses and deviating surfaces for the formation of the image; the pupil, the aperture in the diaphragm; and the retina takes the place of the sensitive plate on which the image of the external object is thrown.

299. *Vision.* By vision is meant the sensation of sight arising from the action on the retina of the molecular energy of the rays of light. Since the image of any point, or the focus corresponding to any radiant, is on the opposite side of the axis of the eye from the radiant, it is evident that the sensation is referred by the mind along the ray in the direction of the radiant. We therefore do not look at the image on the retina, which is in reality inverted, but at the object which is considered erect, although the inverted image is formed on the retina, and can be seen under proper conditions by an observer.

The range of distinct vision for the *normal* eye is from five inches to infinity. The normal eye is capable of so adjusting the lenses which compose it as to make a perfect image on the retina for any position of the object between these limits. The *near-sighted* or *myopic* eye brings parallel rays to a focus, in front of the retina, and, according to the magnitude of this defect, the range of the eye may vary from any distance less than infinity to a distance

less than five inches, say from four feet to four inches. The lenses of the eye in this case possess greater refractive power than those of the normal eye. The defect may be artificially remedied by the use of positive or concave eye-glasses. The *far-sighted* or *presbyopic* eye differs from the normal eye in a lack of power of adjustment for near objects. It brings parallel rays to a focus on the retina equally well with the normal eye, and fails alone in doing so for the near objects, whose images would be formed behind the retina. This defect can be remedied by the use of convex lenses, to assist the lenses of the eye in refracting the rays when near objects are to be viewed.

When the eye has a curvature of the cornea, or of its crystalline lens, which differs in the horizontal and vertical planes, the radiants will be brought to a focus on the focal lines instead of to a point, and this defect is called *astigmatism*. The optical centre of the eye may be taken in the plane of the iris at the centre of the pupil, for all practical purposes relating to the discussion of optical images.

300. ***Optical Images.*** An optical image is an assemblage of foci, which are conjugate to the consecutive radiants on the surface of the object whose image is to be formed. The optical image may, or may not, have a general resemblance to the object. When the deviating surfaces are irregular in form, these images are unlike, and when regular, the image is a picture more or less similar to the object in appearance. Every ray proceeding from a radiant carries with it an image of the radiant, and if a small pencil of these rays be deviated by media bounded by regularly curved surfaces, the focus will present a visible image of the radiant.

For the ordinary optical instruments, it is sufficient to consider only small direct or oblique pencils, and to construct the foci of the salient points of the object, considered as radiants, and thus get the salient points of the image.

301. In Figure 69, let the radiant PQ be a right line perpendicular to the axis of the lens, whose optical centre is at O. The only rays proceeding from P which are met by the lens are those of the cone lPl'. The axial ray of this cone, taken as a small oblique pencil, passing through O is undeviated, and the focus of all rays of this pencil may, without material error, be taken on this line. To find the focus, draw the extreme ray Pl; its deviation by the lens is

represented by the general symbol $\frac{a}{F_2}$, in which a is the distance from the axis to the point in which the ray meets the lens, and F_2 is the principal focal distance, expressed in the same unit of measure. This represents the power of the lens to deviate any ray, and its value for any ray depends therefore on the angle the ray makes with the axial ray before deviation, the curvature of the surfaces

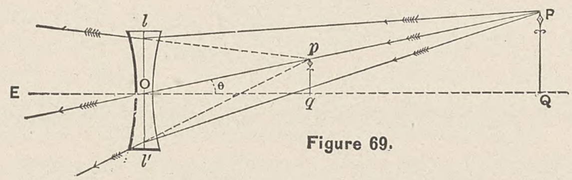

Figure 69.

and index of refraction. The intersection of the rays lp and PpO is the image of P, and all rays from P passing through the lens, produced backward meet at p. This is then the image of P, and similarly q is the image of Q, and pq of PQ. Let θ be the angle made by PO with the axis of the lens; then taking O as an origin, we have

$$\mathrm{OP} = \frac{\mathrm{OQ}}{\cos\theta} = \frac{f}{\cos\theta},$$

in which f represents the distance of the point of the object on the axis of the lens from the optical centre. Substituting for f its value in

$$\frac{1}{f''} = \frac{1}{F_2} + \frac{1}{f}, \tag{361}$$

we have, finally,

$$f'' = \frac{F_2}{1 + \frac{F_2}{f}\cos\theta}, \tag{362}$$

in which f'' is the distance of the focus, measured on the axis of the pencil, and F_2 the principal focal distance of the lens. This equation is similar in form to the polar equation of a conic section, and therefore f'', the distance of the focus of each radiant, measured on the axis of each pencil of rays, is a radius-vector of a conic section, referred to the optical centre as a pole. The excentricity of

the conic section depends on the sign of $\frac{F_2}{f}$, and its value compared with unity. It is, however, sufficiently exact for the practical purposes of illustration to regard the image of PQ similar in all respects to PQ.

302. The equation for discussing optical images for lenses will then be

$$f'' = \frac{\pm F_2}{1 \pm \frac{F_2}{f} \cos \theta}, \tag{363}$$

in which F_2 is negative for convex, and positive for concave lenses.

By transferring the origin for reflectors to the centre of curvature, Eq. (307) readily becomes

$$\frac{1}{f'} = -\frac{2}{r} - \frac{1}{f}, \tag{364}$$

and the corresponding equation for images for reflectors becomes

$$f' = \frac{\mp F_1}{1 \pm \frac{F_1}{f} \cos \theta}, \tag{365}$$

in which F_1 is negative for concave, and positive for convex reflectors.

303. It is necessary now simply to determine whether the image is *real* or *virtual*, *erect* or *inverted*, and *larger* or *smaller* than the object.

1°. *Image Real or Virtual.* Since θ, in the usual cases where this discussion applies, differs but little from 0°, cos θ is positive and nearly unity, and since in lenses positive values of f correspond to real objects, and negative values of f to virtual objects, and positive values of f'' to virtual, and negative values of f'' to real images, we at once draw the following conclusions, viz.,

With a concave lens or a convex reflector:

All real objects give virtual images.

All virtual objects between the lens and $-F_2$ give real images.

All other virtual objects give virtual images.

With a convex lens or a concave reflector:

All real objects beyond $+F_2$ give real images.
All real objects within $+F_2$ give virtual images.
All virtual objects give real images.

2°. *Image Erect or Inverted.* If the image and object are on the same side of the optical centre, the image will be erect; if on opposite sides, inverted, since the axes of all the pencils pass through the optical centre and the object is always considered erect. But, in the former case, f and $f'' \cos\theta$ will have *like* signs, and in the latter *contrary* signs; hence, if the ratio of the distances of the object and image, measured on the axis of the lens, is *positive*, the image will be erect, and if *negative*, it will be inverted.

3°. *Relative Size of the Image and Object.* The linear dimensions of the image and object are the distances of the conjugate foci of the extreme points from the axis of the lens, and these distances are directly proportional to their distances from the optical centre, measured on the axis of the lens. The ratio of f and $f'' \cos\theta$ will therefore determine the relative size of the image and object.

304. ***Concave Lens.*** The equation is

$$f'' = \frac{F_2}{1 + \frac{F_2}{f}\cos\theta}. \qquad (366)$$

1°. Real objects.

f is positive, and hence f'' is positive, but $f'' < f$:
$\therefore$ the image is virtual, because f'' is $+$;
the image is erect, because $\dfrac{f}{f'' \cos\theta}$ is $+$;
the image is smaller, because $f'' \cos\theta < f$.

2°. Virtual objects.

(*a*.) Between the lens and $-F_2$, f is negative and $f'' > f$:
$\therefore$ the image is real, because f'' is $-$;
the image is erect, because $\dfrac{-f}{-f'' \cos\theta}$ is $+$;
the image is larger, because $f'' \cos\theta > f$.

(*b.*) Between $-F_2$ and $-\infty$, f is negative, f'' is positive and until near $-2F_2$, $f'' > f$, after which $f'' < f$:

$\therefore$ the image is virtual, because f'' is $+$;

the image is inverted, because $\dfrac{-f}{+f'' \cos\theta}$ is $-$;

the image is larger, when beyond $2F_2$, because $f'' \cos\theta > f$;

the image is smaller, when within $2F_2$, because $f'' \cos\theta < f$.

305. ***Convex Lens.*** The discussion is similar, and the results are as follows:

Real objects between $+\infty$ and $+2F_2$ give real images, inverted and smaller.

Real objects between $+2F_2$ and $+F_2$ give real images, inverted and larger.

Real objects between $+F_2$ and the lens give virtual images, erect and larger.

Virtual objects give real images, erect and smaller.

Corresponding discussions may be made of the images formed by reflectors, from Eq. (365).

306. The ratio of the visual angle subtended by the image and object will vary with the position of the eye and the relative sizes of their apparent diameters. To obtain an expression for this ratio, let α and α' be the visual angles of the object and image, and Δ the distance of the optical centre of the eye from the optical centre of the lens. Then (Figure 69) supposing E to be the place of the eye, we have

$$\alpha' = p\mathrm{E}q = \frac{pq}{\mathrm{EO} + \mathrm{O}q} = \frac{\mathrm{PQ}}{f} \cdot \frac{f''}{f'' + \Delta}; \tag{367}$$

and since the visual angle of the object is practically the same, whether the eye be taken at E or O, because of its comparatively great distance, we have

$$\alpha = \mathrm{POQ} = \frac{\mathrm{PQ}}{\mathrm{OQ}} = \frac{\mathrm{PQ}}{f}, \tag{368}$$

and hence
$$\frac{\alpha'}{\alpha} = \frac{f''}{f'' + \Delta}. \tag{369}$$

Therefore, the image of an object by a concave lens appears smaller than the object, and since it is on the same side of the lens as the object, it appears erect.

In the convex lens, f'' is negative, and we have

$$\frac{\alpha'}{\alpha} = \frac{-f''}{-f'' + \Delta} = \frac{f''}{f'' - \Delta}. \tag{370}$$

Supposing distinct vision possible for all positions of the eye, we see, supposing Δ to have the values,

$$\Delta = 0, \quad < \frac{f''}{2}, \quad = \frac{f''}{2}, \quad > \frac{f''}{2}, \quad = f'', \quad = 2f'',$$

that α will vary, and its corresponding values will be

$$\alpha' = \alpha, \quad > \alpha, \quad = 2\alpha, \quad > 2\alpha, \quad = \infty, \quad = \alpha.$$

The images in these cases are real, and being on the opposite side of the lens from the object, will appear inverted. Finally, when the object is at the principal focal distance, the emergent rays will be parallel, and the visual angle will be independent of the position of the eye, and the same as if it coincided with the optical centre of the lens.

307. ***Optical Instruments.*** These embrace the various combinations of reflecting and refracting surfaces which form images of objects, in such positions and of such dimensions as are needed for special observations. Dispersion, which generally accompanies refraction, is not considered in the following discussion; its influence in modifying the construction of particular instruments will be referred to subsequently.

308. ***The Camera Lucida*** enables an observer to trace readily a reduced image of an external object. It consists of a quadrangular prism of glass MKNH, Figure 70, whose diedral angles at K and H are 90° and 135°, respectively. The rays proceeding from an external object A are deviated by the positive lens Z, making a the virtual image, from which rays enter the prism

nearly normal to KN; they are then reflected at NH and HM, to enter the pupil of the eye at P with direct rays from a''; the image may be traced by the pencil of the observer at a'', since the rays from the pencil point and image have the same vergency when they enter the eye. The construction and course of the rays are shown in the figure.

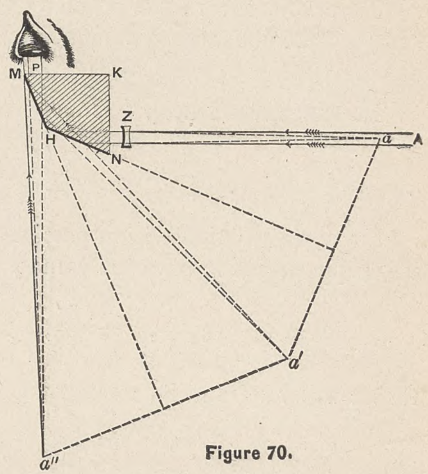

Figure 70.

309. ***The Camera Obscura***, Figure 71, is used to form a real image of an external object on a screen or upon a sensitive photographic plate, placed at the back of a darkened chamber. Since the image is real, the lens is negative, and the relative distances of the object and image from the optical centre are given by Eq. (320). The *magic lantern* is essentially the same in principle. The object in this case being a transparent picture, painted or photographed on

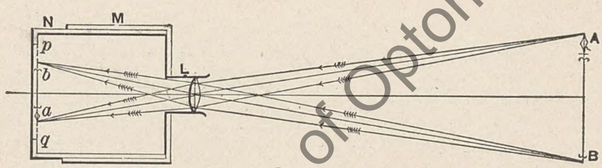

Figure 71.

glass, and strongly illuminated by artificial light, which is concentrated on the object by a condensing lens, and the picture is placed near the principal focus of the projecting lens, in order that the image may be as large as the illumination will permit.

310. ***The Solar Microscope***, Figure 72, the same in principle as the magic lantern, consists essentially of a plane mirror MM', to reflect the solar rays upon a condensing lens C, by means

of which the transparent object PQ is strongly illuminated, and of a short-focus lens L to form an image of PQ. Since the solar light is intense, the image of PQ can be made very large in comparison with it, and yet be very bright. The object PQ is placed beyond

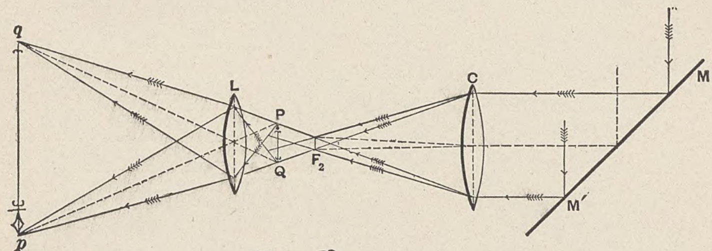

Figure 72.

the focus F_2 of the lens C, to diminish the injurious effect of the heat rays which are collected in or near the focus F_2, and also to intersect the diverging cone of rays, so that all the object will be illuminated.

311. ***Microscopes.*** A *microscope* is an optical instrument used to form a magnified image of a very minute and near object, and to view the image. If the eye possessed the property of distinct vision at all distances, a clear and magnified image would be formed by it as the object approached the optical centre of the eye. But what is gained in magnitude after the object passes the inferior limit of distinct vision is more than lost by the indistinctness arising from the inability of the lenses of the eye to bring the rays to a focus on the retina. The *simple microscope* is a convex lens, whose principal focal distance is less than the inferior limit of distinct vision. Such a lens, whose principal focal distance is five inches, would be a microscope for an eye whose inferior limit is six inches, but would not be for one whose limit is four inches. When the object, as in Figure 73, is placed nearer the lens than F_2, the image is *virtual, erect,* and *larger* than the object. When the object is placed at F_2, the emerging rays of each pencil are parallel to the undeviated ray, and the apparent size of the image is just the same as if the optical centre of the eye were placed at the optical centre of the lens.

312. ***The Magnifying Power*** is the ratio of the apparent diameter of the image to that of the object, the object and image being supposed properly placed for distinct vision.

But since objects viewed by the microscope can be placed at any convenient distance from the eye, it is usual to compare their apparent diameters with those of their images when they are at the inferior limit of distinct vision, because their details can best be observed at this location. If we suppose the object to be so placed that the image formed by the simple microscope is at the limit of distinct vision from the eye, then the direct ratio of the diameters of the image and object, since they are compared at the same dis-

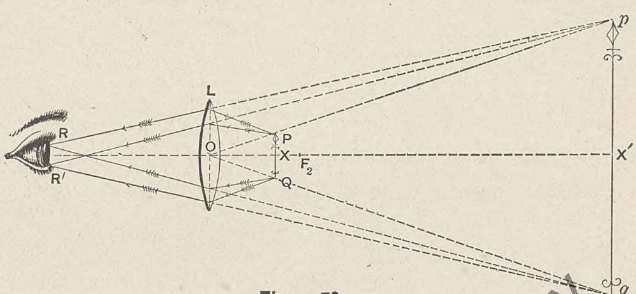

Figure 73.

tance, will be the measure of the *magnifying power* of the microscope. Representing the magnifying power by M, we have then, from Figure 73,

$$M = \frac{OX'}{OX} = \frac{f''}{f}. \qquad (371)$$

Letting d represent the limit of distinct vision, and Δ the distance of the eye from the lens, we have

$$f'' = d - \Delta,$$

and

$$\frac{1}{f''} - \frac{1}{f} = -\frac{1}{F_2};$$

whence,

$$M = 1 + \frac{d - \Delta}{F_2}. \qquad (372)$$

When the object is at the principal focus, the image appears of

the same size to the eye as if the latter were at the optical centre of the lens, and we have

$$M = 1 + \frac{d}{F_2}. \tag{373}$$

The same value results when Δ is taken so small compared with d as to be negligible.

When F_2 is so small compared with d that unity may be omitted with respect to $\frac{d}{F_2}$, we have, approximately,

$$M = \frac{d}{F_2}. \tag{374}$$

For a particular observer, the utility of the microscope is determined by the greatest visual angle under which it enables him to see clearly any definite length, whatever may be the angle under which he sees it without the lens.

Thus, if the millimetre be the unit of length, the visual angle without the microscope would be $\frac{1}{d}$, and with it would be $\frac{M}{d}$; therefore, the advantage would be, from Eq. (372),

$$\frac{M}{d} = \frac{1}{d} + \frac{1 - \frac{\Delta}{d}}{F_2}; \tag{375}$$

or approximately as before, calling P the useful power of the microscope,

$$P = \frac{M}{d} = \frac{1}{d} + \frac{1}{F_2}, \tag{376}$$

which shows that a myopic eye has an advantage over a presbyopic or normal eye, unless F_2 is so small as to make this difference negligible.

313. ***The Field of View*** of the microscope is the angular space embraced by the eye within which spherical aberration does not appreciably affect the clearness of vision; it is experimentally found to be within 9° or 10° around the principal axis of the microscope. But although the magnifying power of the microscope

increases when the radii of curvature of the surfaces diminish, the spherical aberration increases also.

To diminish this effect, the field of view must be correspondingly contracted, by means of a diaphragm which stops out the marginal rays.

314. *Oculars.* Simple microscopes formed of two lenses are generally used to view images formed by optical instruments, and of these it is sufficient to describe two, viz., the *positive,* in which the image is formed in front of the lenses of the ocular, and the *negative,* in which it is formed between the lenses.

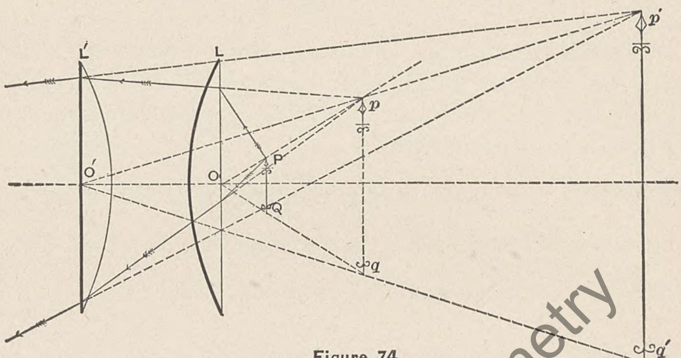

Figure 74

1°. *Positive Ocular.* Let the two plano-convex lenses L and L′ (Figure 74) be placed with their convex surfaces turned towards each other, and let PQ be the object to be viewed. The image of PQ formed by L is pq, and the image of pq formed by L′ is $p'q'$, as is readily seen from the figure.

We have from L, with the origin at O,

$$\frac{1}{f''} - \frac{1}{f} = -\frac{1}{F_2}, \qquad (377)$$

and from L′, with O′ as an origin, representing the distance OO′ by D, and the limit of distinct vision by d,

$$\frac{1}{f_1''} - \frac{1}{f'' + D} = -\frac{1}{F_2'}. \qquad (378)$$

The magnifying power of the ocular is

$$M = \frac{p'q'}{\mathrm{PQ}} = \frac{p'q'}{pq} \times \frac{pq}{\mathrm{PQ}} = \frac{d}{D + f''} \cdot \frac{f''}{f}; \tag{379}$$

but from Eqs. (378) and (379) we have, since $f_1'' = d$,

$$\frac{d}{D + f''} = 1 + \frac{d}{F_2'},$$

and

$$\frac{f''}{f} = 1 + \frac{f''}{F_2} = 1 + \frac{\frac{f_1''F_2'}{f_1'' + F_2'}}{F_2} - \frac{D}{F_2};$$

$$\left.\begin{aligned} \therefore\ M &= \left(1 + \frac{d}{F_2'}\right)\left(1 + \frac{f_1''F_2'}{F_2(f_1'' + F_2')} - \frac{D}{F_2}\right) \\ &= 1 + \frac{d - D}{F_2} + \frac{d}{F_2'} - \frac{dD}{F_2F_2'}, \text{ approx.} \end{aligned}\right\} \tag{380}$$

When $F_2 = F_2'$ and $D = \frac{2}{3}F_2$, the ocular is called Ramsden's, and we have for the magnifying power,

$$M = \frac{F_2 + 4d}{3F_2}, \tag{381}$$

and when F_2 is small compared to d, equal to

$$M = \frac{4}{3} \cdot \frac{d}{F_2}. \tag{382}$$

2°. The *negative* or *Huyghen's* ocular consists of two convexo-plane lenses. The images formed by the two lenses L and L′ of this ocular are shown in Figure 75; thus the converging pencil of rays would form the object PQ, were they not met by the lens L; but being deviated by it, they form the real image pq, which the lens L′ magnifies into $p'q'$.

A discussion similar to that for the positive ocular gives for the magnifying power,

$$M = \left(1 + \frac{d}{F_2'}\right)\left(1 + \frac{dF_2'}{F_2(d + F_2')} - \frac{D}{F_2}\right), \tag{383}$$

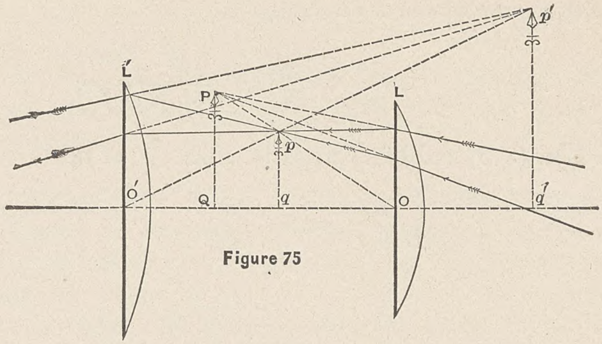

Figure 75

identical in all respects. Making $F_2' = \frac{1}{3}F_2$ and $D = \frac{2}{3}F_2$, we have

$$M = \frac{6d + F_2}{3F_2} \quad \text{or} \quad \frac{6d}{3F_2}, \text{ approximately.} \tag{384}$$

315. ***The Compound Microscope*** consists essentially of an objective O, which is a simple microscope formed of one or several lenses, whose purpose is to form a real, magnified, and inverted image of a minute object, and an ocular E, also a simple microscope, to magnify and view this image. Figure 76 indicates the

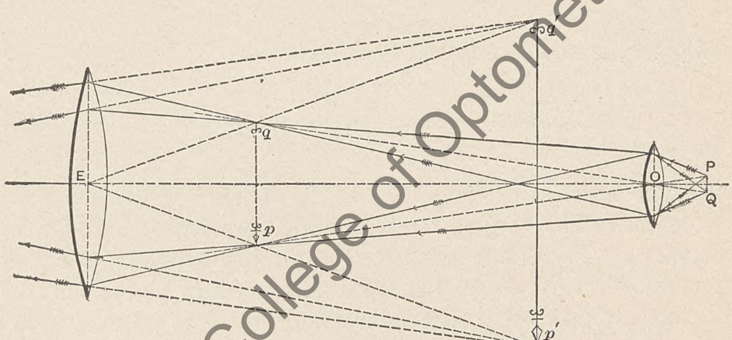

Figure 76

size and location of the image corresponding to the object PQ. If the distance of the ocular from the objective can be varied, the ocular can be placed always in such a position that the image formed by it will be at the inferior limit of distinct vision from the eye,

wherever the image of the objective may be. In this case, however, the magnifying power, which depends on the ratio of $p'q'$ to PQ, will vary with the position of the ocular with respect to the objective.

Therefore, in order to keep a constant magnifying power, this distance should be invariable, and the tube containing them may be moved towards or from the object. The magnifying power is evidently

$$M = \frac{p'q'}{\mathrm{PQ}} = \frac{p'q'}{pq} \times \frac{pq}{\mathrm{PQ}}. \tag{385}$$

The first factor is the magnifying power of the ocular, and the second is that of the objective.

316. *Telescopes.* A telescope is an optical instrument composed of an object-glass or reflector to form a real image of a real and distant object, and of an ocular to magnify and view the image. Therefore, the object-glass must be essentially convex, if the telescope is a refractor, and if a reflector, the object mirror must be concave ; the ocular may be either concave or convex.

1°. The refracting telescopes are the *astronomical*, the *Galilean*, and the *terrestrial*.

317. *The Astronomical Refracting Telescope,* Figure 77, consists essentially of a convex lens L, called the *field lens*

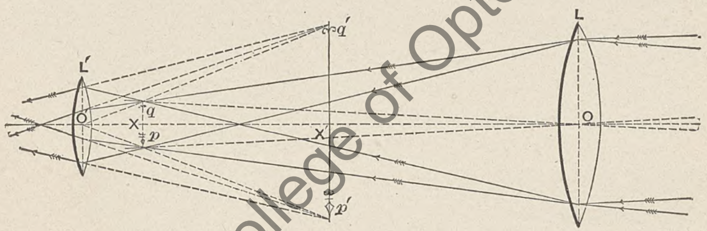

Figure 77

or *objective*, and a convex lens L′, called the *ocular* or *eye lens*. The incident pencil of rays from the upper point of the object is brought by the objective to a focus at p, and the rays of the pencil from the lower point are united at q ; pq is therefore a real image whose

linear magnitude is to that of the object directly as their distances from O. From p a pencil of rays proceeds which is met by the eye lens L′, and so deviated in the figure as to have the virtual focus of p at p'; p' is determined by drawing the ray through O′ and producing it backward; it is the axial ray of the small pencil and passes through O′ undeviated; any other ray from p will be deviated, and when produced backward will meet pO′ in p', which is the virtual image of p. The image of any point, as q, is found in the same manner, and thus $p'q'$ is determined. If the lens L′ were drawn out until the image pq occupied its principal focus, the pencils from pq after deviation would be composed of parallel rays, each being parallel to the axial ray of its small pencil. For the normal or myopic eye, O′X is always slightly less than the principal focal distance of the ocular, but for an infinitely presbyopic eye it is equal to it. In determining the magnifying power of telescopes, the distance O′X is usually taken equal to F_2', or the principal focal distance of the ocular. When the object is so far off that the rays of each pencil may be taken as parallel, the image pq is formed in the principal focus of the objective, and the length of the telescope is then equal to $F_2 + F_2'$. The objective is usually formed of two lenses, the outer one a double convex and the inner a double concave. The adjacent surfaces of the two lenses have exactly equal and opposite curvatures, and generally are in actual contact, and the curvature of the inner surface of the concave lens is much less than that of the outer surface. The forms and refractive powers of the lenses are such as to satisfy, as far as possible, the conditions of correcting both spherical aberration and color.

318. The principal oculars used in the astronomical telescope are Huyghen's or the negative, and Ramsden's or the positive ocular. The former is called the achromatic eye-piece, because the correction for color is nearly fulfilled. It is suitable for examining the physical features of celestial objects, and is not well adapted for taking observations which require measures of time or space, since the image is formed between its lenses. For such observations, the observer must be able to see distinctly fine threads or spider-lines stretched across a stop fixed in the telescope-tube, in coincidence with the optical image of the celestial object. This condition is fulfilled in Ramsden's ocular, since the image formed by the object-

glass is situated close to the inner lens of the ocular, on the side towards the object-glass, as shown in Figure 74.

319. If lines be drawn from the opposite extremities of the object and eye-lens, as shown in the figure, the pencils of light emanating from points of the image included between these lines are incident on the eye-lens. Hence these points will appear of equal illumination, while the points exterior to these limits will exhibit decreasing brightness. This exterior ring is known as the *ragged edge* of the field of view. A diaphragm is inserted in the eye-tube to cut off the partial pencils and leave the whole pencils alone, to pass to the eye-lens. This area appears uniformly bright, and is called the *field of view*. The diaphragm is placed in the plane of the image, and to it are attached the wires or spider-lines referred to above. In telescopes such as the transit, when it is desirable to have a more extended field of view, the eye-piece has a motion perpendicular to the axis of the instrument, so that the whole field in this case may be considerably extended laterally, in order to enable the observer to notice the passage of a celestial body over several wires.

320. ***The Galilean Telescope,*** Figure 78, commonly known as the opera or reconnoitring glass, has a *convex* lens for an

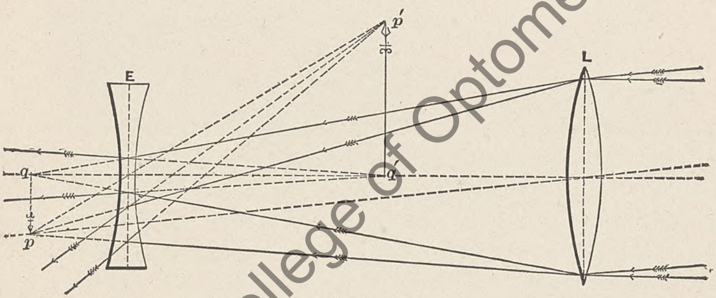

Figure 78

objective L, and a *concave* lens E for the eye-lens. The image *pq* of a distant object PQ would be formed by the objective as in the preceding telescope, were not the pencils intercepted by the eye-lens E; *pq* is then a virtual object for E, and its image is determined as indicated by the construction lines in the figure. The lens E is

usually taken to be at its principal focal distance from pq, and when the incident rays on L are parallel, the length of the telescope is equal to $F_2 - F_2'$. This instrument is not adapted to use in astronomical measurements. The image of PQ, as seen in the telescope, is erect.

321. ***The Terrestrial Telescope,*** Figure 79, is nothing but the astronomical telescope, L, E, with the addition of two convex lenses l, l', called the erecting-piece. The first image pq is formed at the principal focus of l, the pencils of which after deviation are parallel; these parallel beams are brought to a focus by l' at its principal focal distance, and form the second image $p'q'$. The eye-lens E deviates the pencils proceeding from $p'q'$, making them appear to come from the virtual image $p''q''$. The construc-

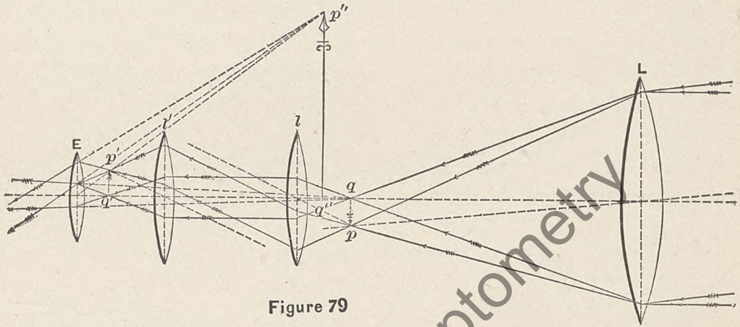

Figure 79

tion of these images is shown in the figure. The introduction of the erecting lenses is attended with loss of light, and hence would be unwarrantable in an astronomical telescope. Besides, since celestial objects are generally round, no sense of inversion is experienced in viewing their images, although it is necessary to remember the fact that the lower part of the image corresponds to the upper part of the object, and the right of the image to the left of the object, and *vice versa*. The lenses of the erecting-piece in the terrestrial telescope are usually made of the same power, and it therefore has no magnifying influence.

322. 3°. The principal *reflecting telescopes* are named after their inventors, and are the *Herschelian*, the *Newtonian*, the *Grego-*

rian, and the *Cassegrainian*. In all these, since it is necessary to form a real image of a real object, the object reflector must be concave. These reflectors are usually made of speculum, and have an aperture much larger than it has been possible to give to the refractors.

Herschel's telescope is the simplest and most commonly used of the reflectors. It consists of a concave speculum, ground to the form of a paraboloid, whose focus is near the mouth of the tube; it is slightly inclined to the axis of the tube, in order that the image may be thrown near one side, so that the head of the observer may intercept as little light as possible. The image is viewed by an ocular, which is a simple microscope.

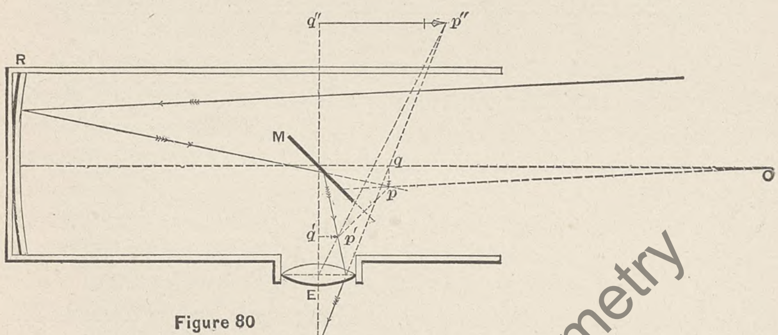

Figure 80

323. ***The Newtonian Telescope,*** Figure 80, consists of the concave spherical reflector R whose centre of curvature is at O, the plane reflector M, and the ocular E. To construct the point p'' of the image, draw any ray, and find its intersection p after deviation with the ray proceeding from the point P through the optical centre O of the object reflector. This ray being intercepted by the plane mirror M, the point p is a virtual radiant for M, and its real focus is p', which in turn is a real object for E, giving a virtual image p''. Other points of the image being similarly constructed, we have $p''q''$ at the limit of distinct vision for the eye behind the lens E.

324. ***The Gregorian Telescope,*** Figure 81, has for its essential parts the object-reflector R with centre at O, the small

concave reflector R$'$ with centre at O$'$, and the eye-lens E. To construct the point p'' of the image, draw any ray, as the extreme ray, for example; its intersection with the undeviated ray through

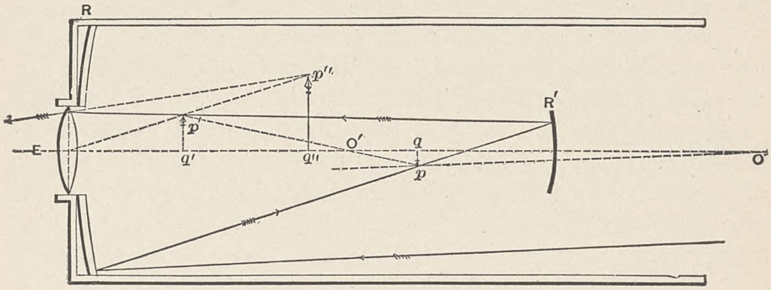

Figure 81

O determines p, and the image pq is real. From p draw any ray to the small reflector; its intersection with the undeviated ray through O$'$ determines p', and thus the image $p'q'$ of pq by the small reflector; the virtual image of $p'q'$ by the eye-lens E is $p''q''$, as in previous cases. The image as seen by this telescope appears erect. The small mirror R$'$ is capable of being moved along the axis by a rod and screw fixed along the bottom of the telescope tube. It is apparent that as it approaches pq, the image $p'q'$ moves towards E, and conversely.

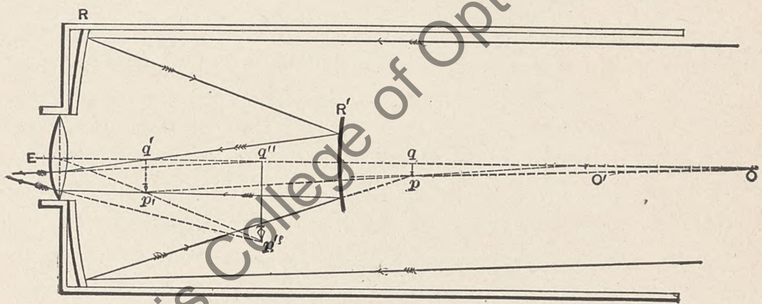

Figure 82

325. ***The Cassegrainian Telescope,*** Figure 82, is similar to the Gregorian, except that the mirror R$'$ is convex. There-

fore the image pq must be a virtual object for R′, in order that a real image $p'q'$ be formed in front of the eye-lens E. The construction of the image $p''q''$ is indicated in the figure, and is analogous in every respect to that of the preceding figure. Objects seen through this instrument appear inverted.

326. ***The Magnifying Power*** of any combination of mirrors or lenses is the ratio of the apparent magnitude of any linear dimension of an object viewed by the eye directly, to the apparent magnitude of the corresponding linear dimension of the image as seen by means of the combination. We have seen, Art. (303), that the linear dimension of an object is to the linear dimension of its image formed by a lens or a mirror directly as their distances from the optical centre of the lens or mirror.

Let n be the number of lenses and mirrors, D the distance of the object from the objective, and d the distance of the eye from the ocular; let I_1, I_2, I_3, ...I_n, represent the successive images formed; and

f_1, f_2,	be the distance of		I_1	from the	1st and 2d	lenses or	mirrors,
f_3, f_4,	"	"	I_2	"	2d and 3d	"	"
....	"	"	....	"	....	"	"
f_{2n-3}, f_{2n-2},	"	"	I_{n-1}	"	$(n-1)^{th}$ and n^{th}	"	"
$f_{2n-1}+d$	"	"	I_n	"	eye;		

and let L be the linear dimension of the object. We will have then,

$$L\frac{f_1}{D}, \text{ the linear magnitude of } I_1;$$

$$L\frac{f_1}{D}\cdot\frac{f_3}{f_2}, \quad \text{"} \quad \text{"} \quad I_2;$$

$$\text{........} \quad \text{"} \quad \text{"} \quad \ldots$$

$$L\frac{f_1\cdot f_3\cdot\ldots\cdot f_{2n-1}}{D\cdot f_2\cdot f_4\cdot\ldots\cdot f_{2n-2}} \quad \text{"} \quad \text{"} \quad I_n.$$

Dividing this last expression by $f_{2n-1}+d$, the distance of I_n

from the eye, we have for the apparent angular magnitude of the last image,

$$\alpha' = \frac{L}{D} \cdot \frac{f_1 \cdot f_3 \cdot f_5 \cdot \ldots \cdot f_{2n-3} \cdot f_{2n-1}}{f_2 \cdot f_4 \cdot f_6 \cdot \ldots \cdot f_{2n-2} \left(f_{2n-1} + d\right)}. \tag{386}$$

If C be the distance at which the object is seen distinctly by the eye of the observer, the apparent angular magnitude of the object seen without the combination is

$$\alpha = \frac{L}{C} = \frac{L}{D} \cdot \frac{D}{C}. \tag{387}$$

The ratio of these apparent angular magnitudes, and hence the magnifying power of the combination, is

$$M = \frac{\alpha'}{\alpha} = \frac{C}{D} \cdot \frac{f_1 \cdot f_3 \cdot f_5 \cdot \ldots \cdot f_{2n-3} \cdot f_{2n-1}}{f_2 \cdot f_4 \cdot f_6 \cdot \ldots \cdot f_{2n-2} \left(f_{2n-1} + d\right)}. \tag{388}$$

If the object be a heavenly body,

$$\frac{C}{D} = 1,$$

and the magnifying power, therefore, for any astronomical telescope is

$$M = \frac{f_1 \cdot f_3 \cdot \ldots \cdot f_{2n-3} \cdot f_{2n-1}}{f_2 \cdot f_4 \cdot \ldots \cdot f_{2n-2} \left(f_{2n-1} + d\right)}. \tag{389}$$

If the final image is seen by pencils of nearly parallel rays, then $\dfrac{f_{2n-1}}{f_{2n-1} + d}$ is approximately unity, and as this is the usual condition, we have finally,

$$M = \frac{f_1 \cdot f_3 \cdot \ldots \cdot f_{2n-3}}{f_2 \cdot f_4 \cdot \ldots \cdot f_{2n-2}}. \tag{390}$$

327. If pencils from points of the object and corresponding points of the image are nearly perpendicular to the lenses or mirrors of the combination in their passage, their breadths are directly proportional to the distances of the lenses or mirrors from the intermediate foci. Then, if A be the breadth of a pencil falling on the object-glass or mirror, we will have

$A \dfrac{f_2}{f_1}$, the breadth on the 2d lens or mirror;

$A \dfrac{f_2}{f_1} \cdot \dfrac{f_4}{f_3}$, " " 3d " "

. . . . " " . . " "

$A \dfrac{f_2 \cdot f_4 \cdot \ldots \cdot f_{2n-2}}{f_1 \cdot f_3 \cdot \ldots \cdot f_{2n-3}}$, " " n^{th} " "

$e = A \dfrac{f_2 \cdot f_4 \cdot \ldots \cdot (f_{2n-1} + d)}{f_1 \cdot f_3 \cdot \ldots \cdot f_{2n-3} \cdot f_{2n-1}}$, " of the pencil entering the eye. (391)

Substituting in the expression for the magnifying power, we have

$$M = \frac{C}{D} \cdot \frac{A}{e}, \tag{392}$$

and for astronomical telescopes,

$$M = \frac{A}{e}. \tag{393}$$

328. In all telescopes except the Galilean, the size of the pencils is determined by the aperture of the object-glass or spherical mirror, and A will be the value of its diameter. If e be greater than the diameter of the pupil of the eye, some of the light of each pencil is lost, and the effect is the same for the same magnifying power as if the diameter of the object-glass had been decreased. The inferior limit of the magnifying power with a given aperture A is therefore $\frac{A}{e'}$, e' being the diameter of the pupil. If the diameter of the pencil e entering the eye is less than that of the pupil, the image loses distinctness from loss of light, and to remedy this it would be necessary to increase the diameter of the object-glass.

The quantity e is practically the diameter of the image of the object-glass formed by the eye-glass at the position of the eye. If, therefore, the telescope be adjusted to distinct vision, with the eye-piece attached whose magnifying power is to be determined, and the diameter of this image be measured, the quotient obtained by dividing the aperture of the object-glass by this diameter will be the

magnifying power of the telescope. The instrument designed to make this measurement is called the *Dynameter*.

329. *Brightness of Images.* Let B be the brightness of an object, S its surface, and d its distance. Then the quantities of light falling on the unit area, on the pupil of the eye, and on the object-lens or mirror, are

$$B\frac{S}{d^2}, \quad B\pi p^2\frac{S}{d^2}, \quad \text{and} \quad B\pi a^2\frac{S}{d^2},$$

respectively, p and a being the radii of the pupil and aperture. After passing the ocular, supposing no light lost by absorption, and the radius of the ocular ring to be r, the quantity of light received by the pupil will be $B\pi a^2\frac{S}{d^2}\cdot\frac{p^2}{r^2}$, which is less than the quantity of light passing through the ocular, in the proportion of $\frac{p^2}{r^2}$. On the other hand, the apparent surface of the object is increased in proportion to $\frac{a^2}{r^2}$, and hence becomes $\frac{S}{d^2}\cdot\frac{a^2}{r^2}$, which may be placed equal to $\frac{S'}{d^2}$; then the light received by the pupil may be expressed as $B\pi p^2\frac{S'}{d^2}$, and the brightness B is unchanged. If, however, the ocular ring is smaller than p^2, the whole quantity of light $B\pi a^2\frac{S}{d^2}$ enters the eye. This expression may be written

$$\pi\left(\frac{Br^2}{p^2}\right)\left(\frac{Sa^2}{d^2r^2}\right)p^2 = \pi\left(\frac{Br^2}{p^2}\right)\frac{S'}{d^2}p^2;$$

whence we see that the brightness is now $\frac{Br^2}{p^2}$, less than before in the proportion of $\frac{r^2}{p^2}$. Therefore, when the ocular ring becomes less than the area of the pupil, the brightness decreases and the image loses distinctness. This does not apply to a star, because, having no disk, it cannot be magnified, and therefore the larger the aperture of the telescope, the greater the brightness of the image. To study the details of objects having appreciable disks, less powerful oculars should be used than those used for stars. In the best

achromatic telescopes, but 85% of the light is transmitted, and hence objects appear brighter to the naked eye than through the telescope.

330. ***Magnifying Power of Telescopes.*** Applying the general expression, Eq. (390), for the magnifying power, we have, for the

Astronomical refractor, $$\frac{+f_1}{-f_2} = -\frac{F_2}{F_2'}; \qquad (394)$$

Galilean refractor, $$\frac{+f_1}{+f_2} = \frac{F_2}{F_2'}; \qquad (395)$$

Terrestrial refractor,

$$\frac{+f_1}{-f_2}\cdot\frac{+f_3}{+f_4}\cdot\frac{+f_5}{-f_6} = +\frac{F_2}{F_2'} \quad \text{or} \quad \frac{f_1}{f_6}. \qquad (396)$$

When $f_2 = f_5$, and in which $f_3 = f_4$,

Herschelian, $$\frac{+f_1}{-f_2} = -\frac{F_1}{F_2'}; \qquad (397)$$

Newtonian, $$\frac{+f_1}{-f_2} = -\frac{F_1}{F_2'}; \qquad (398)$$

Gregorian, $$\frac{+f_1}{-f_2}\cdot\frac{+f_3}{-f_4} = \frac{F_1}{F_2}\cdot\frac{f_3}{f_2}; \qquad (399)$$

Cassegrainian, $$\frac{+f_1}{+f_2}\cdot\frac{+f_3}{-f_4} = -\frac{F_1}{F_2}\cdot\frac{f_3}{f_2}. \qquad (400)$$

From these results we see that the *Galilean*, *terrestrial*, and *Gregorian* give images *erect*, and the remainder give images *inverted*, as indicated by the essential sign of the expression for the power.

PHYSICAL OPTICS.

331. Heretofore, we have considered the luminous ray as a geometrical right line, and have discussed its changes of direction in reflection and refraction without regarding its physical properties. We now proceed to explain, by the principles of the undulatory theory, the nature of the luminous ray as exhibited by the various

phenomena arising from the action of rays on the organ of vision and on each other, and in their deviation by different media.

332. *Color.* When a small parallel beam of sunlight is intercepted by a prism, a separation of its elements is effected, and each ray of the white light is found not to be homogeneous, but to consist of an innumerable number of other rays, each of which has the property of so affecting the retina as to produce in our minds the sensation of *color*. And generally all natural or artificial light is similarly but variously compounded. When a small beam of homogeneous light, as *red*, for example, falls on a prism placed in its position of minimum deviation, the refractive index for this color and particular prism is readily obtained from Eq. (301),

$$\mu_r = \frac{\sin\left(\frac{\delta + \alpha}{2}\right)}{\sin\frac{\alpha}{2}}. \tag{401}$$

Should the beam consist of two kinds of homogeneous light, δ and hence μ will differ for each. In the solar beam, the deviation of the constituent homogeneous elements varies by continuity from the red to the violet, and therefore each simple ray will have a particular refractive index. When the beam is intercepted by a screen after deviation, a colored image called the *solar spectrum*, whose width is uniform for a given position of the screen, is pictured upon it. Its length, however, depends on the refracting angle of the prism, and on the material of which it is made, as well as on the position of the screen. The colors pass by insensible transition from red to violet, in the order of

Red, Orange, Yellow, Green, Blue, Indigo, and *Violet,*

red being deviated towards the base of the prism the least, and *violet* the most. Each colored ray has a particular refrangibility, and is simple, for it suffers no farther decomposition when deviated again by another prism.

333. This spectrum is, in reality, made up of the overlapping colored images of the aperture through which the solar beam passes, but its colors are impure, since the rays which enter the eye from any part between its extremities proceed from two or more of the

colored images. To obtain a *pure spectrum*, the light is admitted through a narrow aperture or slit, about 3 or 4 mm. in width, with parallel edges, and allowed to fall on a prism placed in the position of minimum deviation, having its refracting edge parallel to the slit. To project this spectrum, an achromatic lens of considerable focal length is placed behind the prism, at double its principal focal distance from the slit. The screen is placed normal to the mean deviated ray, at double the principal focal distance of the lens, and the spectrum will then be rectangular in shape, sharply defined, and will consist of the contiguous colored images of the aperture, arranged according to their refrangibility.

334. *Dispersion.* This property, by which *refractive* media separate white light into its homogeneous elements, is called *dispersion.* All refractive media do not possess this property in the same degree, as is readily shown by using prisms of the same refracting angle, but of different material. The *coefficient of dispersion* is the difference between the refractive indices of the extreme violet and extreme red ray, or $\mu_v - \mu_r$. The *dispersive power* of a medium is measured by the ratio of the excess of the index of refraction of the mean ray over unity to the coefficient of dispersion, or

$$\frac{\mu_v - \mu_r}{\mu_g - 1}. \tag{402}$$

To definitely fix the value of the dispersive power, μ_r, μ_g, and μ_v are taken to be the refractive indices of rays coincident with certain characteristic lines of the spectrum, C, E, and G, and which therefore now fix the wave lengths of the particular red, green, and violet, considered in this measure.

335. If the aperture be made sufficiently narrow, the resulting solar spectrum will be crossed transversely with numerous dark lines, indicating the absence or partial obscuration of certain colored rays of particular refrangibility. Frauenhofer, who in the early part of this century noted nearly 600, and determined the position of 354 of them, called them the *fixed lines* of the solar spectrum ; but they are now generally known as Frauenhofer's lines. He selected eight of the principal lines as standard means of reference, and designated them by the letters A, B, C, D, E, F, G, and H.

Their position, with that of two intermediate lines *a* and *b*, are shown in the normal spectrum, Figure 83. Owing to their fixity of position and sharpness of delineation, they may readily be observed, and the refrangibility of any ray can be determined by their assistance with the greatest accuracy.

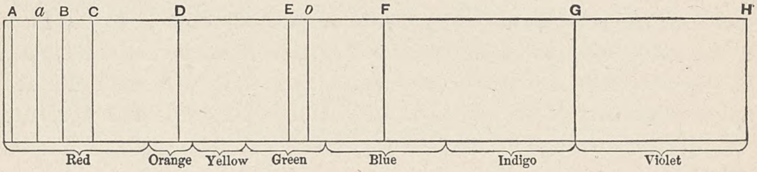

Figure 83.

336. As has been before stated, the dispersion produced by various media is different, and therefore the widths of the color spaces and distances separating the fixed lines would not be invariable in comparing spectra produced by different prisms. But in the *normal spectrum* produced by *diffraction*, the position of the lines and colors depends on the corresponding wave lengths, and hence affords a reliable and fixed standard of reference.

337. The index of refraction μ, being equal to the ratio of the velocities of the ray before and after refraction, and since it differs for each color, we will now see what conclusion can be drawn from the expression for the velocity of wave propagation in any medium, as given by Eq. (119), of Art. 95,

$$V^2 = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4} + \frac{d}{\lambda^6} + \text{etc.} \qquad (403)$$

In this, *a, b, c*, etc., were shown to be constants depending on the constitution and nature of the medium; therefore, for a particular medium, V must vary with λ. This, together with the facts of dispersion, indicates that λ differs for each ray of homogeneous light and is the basis of the explanation offered by the undulatory theory for the unequal refrangibility of the elements of white light. If white light is not dispersed by any medium, we must conclude either that the wave lengths of rays of different colors are equal, or that they are transmitted through this medium with equal velocities; and in the latter case attribute to such a medium a constitution compatible with the observed facts, by properly modifying the

hypotheses which influenced the deductions based upon them. Many independent experiments and observations completely establish the fact of a variation in the wave length due to a variation in color, and therefore the supposition of a constancy of wave length must be abandoned. The most careful observations on the light of the fixed stars and other heavenly bodies have failed to indicate the slightest dispersive power of the intervening media which transmit their light to the earth, and therefore we are forced to conclude that the velocity of transmission is the same for all rays in the interplanetary ether. The inference drawn from this constancy of V is that the ether molecules of space are separated from each other by distances which are practically insignificant, compared to the lengths of the shortest light waves. The analogous supposition made with respect to the wave lengths of sound and the distance apart of the adjacent air-particles, in Art. 183, resulted in the theoretical deduction of the constancy of V for all sound waves of greater or less wave length in air, which deduction is found practically confirmed by observation.

338. Again, since the absolute refractive index of air is only 1.000294, and those of other gases are not greatly in excess of this, we commit no sensible error in assuming the dispersive power of air and other gases to be negligible compared with ordinary solid and liquid media, and can therefore practically assume the relative refractive indices of these media, referred to air as their absolute indices, making the correction in particular cases, when necessary.

339. If two prisms of different material and same refracting angle be used to form the solar spectrum, the widths of spaces occupied by the same color, and the distances apart of the characteristic lines, will not be the same in each, even if the spectra be of the same length. This is called *irrationality of dispersion.* Assuming that the velocity of wave propagation in air is the same for all luminous rays, and remembering that the time during which the motion is propagated over the wave length is the *periodic* time, then either the length of the wave undergoes a change when homogeneous light enters a dispersive medium, or the number of vibrations per second corresponding to this color varies. Certain observations indicate that the number of waves by which a particular color impression is made on the retina is invariable, and there-

fore we conclude that the wave lengths change on their entrance into such a medium. The wave length of any color for any medium will then be found by dividing the wave length in air by the refractive index of this color for that medium. The method of finding the wave length in air will be described in the subject of *diffraction*, and when once determined, together with the velocity of light in air or vacuo, we can find the number of vibrations corresponding to any color of the normal spectrum, which therefore becomes the *constant for this color*.

340. ***Color*** may therefore be defined to be the sensation arising in the mind, due to the effect of a certain number of waves of light affecting the retina per second of time. We have in sound the analogous property of *pitch*, and it is a matter of common observation that the pitch of a sound due to a definite number of vibrations per second may vary through a considerable range, *as heard by the ear*, rising and suddenly falling, as the observer rapidly approaches and recedes from the sounding body. So, likewise, it is theoretically possible to have a color rise and fall in the scale, provided a sufficient change be made in the number of light vibrations which enter the eye in a given interval, although the rays from a luminous source may themselves have but a single wave length or periodicity. The application of this physical fact is of use in the spectroscopic study of the solar prominences.

341. Beyond the extremities of the luminous spectrum, *ultra-violet* rays of greater, and *ultra-red* rays of less refrangibility exist, which are however not visible. The existence of the former is made known by their *actinic* or chemical effect, and of the latter by their *calorific* action. They are essentially of the same nature as the luminous rays, differing only in their refrangibility, and necessarily therefore in their wave lengths. Observation shows in reality that each ray of the visible spectrum possesses heat, light, and actinic properties, differing only in degree, and if we consider the spectrum extended to include the ultra rays, the heating effect is a maximum in the ultra red, the luminosity in the yellow, and the actinic in the ultra violet. These properties are more or less modified by the absorptive powers of the media through which the propagations pass.

342. ***Absorption.*** The undulation conveys kinetic energy from the luminous source to the ether of other matter. Each radiation possesses certain properties, exhibited in the complex effects of heat, light, and chemical action, and these properties are found to vary with λ. Since the intensity of the radiation depends on mv^2, all the effects which the radiation produces must vary with this energy, and we will see that these effects are also proportional to the periodicity of vibration. In sound, for example, the intensity varies with mv^2, and the property of pitch, which is appreciated by sensation, depends on λ. Now a careful examination of each radiation throughout the visible and invisible normal spectrum discloses the fact that each radiation has one or more of the properties of *heat, light,* or *chemical action,* depending on its wave length or periodicity; and since these effects are different in their nature, we have no means of referring them to a common standard, and by subsequent combination get at the true measure of the energy proper to each radiation. If we refer them to sensation only, the last is inappreciable; if we disregard sensation, the second is omitted; if we consider mechanical work only, since heat has its equivalent in work, it would appear that this would be the true measure for the intensity of the radiation. A careful study of the normal spectrum enables us to ascertain how much of each or all of these properties belong to each radiation.

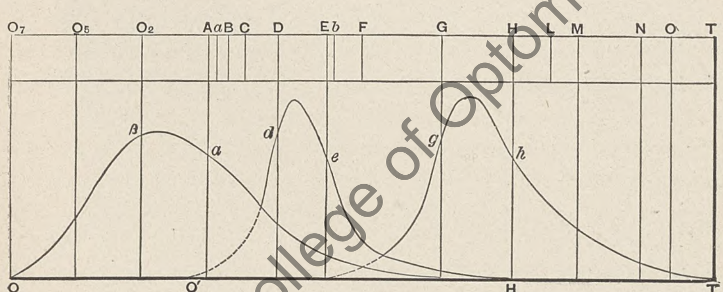

Figure 84.

343. Let us suppose that the ordinates of the curve O*βa*, Figure 84, represent the relative intensities of the heat in each radiation, whose abscissa marks its place in the spectrum, the curve O′*de*H the relative intensity of the light, and the curve *gh*T the

relative intensity of the chemical action. Then each radiation has, as we see, a complex effect, and the individual effects of light, heat, and chemical action of a radiation are absolutely inseparable. They cannot be dissociated by prismatic refraction, since the ray has but one value for its refractive index, and it obeys the law of Snell or Descartes; nor by absorption, for, as will be shown, all absorbing media act alike on each. Hence, in all cases, each property of any given ray is found with its same specific value as those of the other two which accompany it in the spectrum. The philosophical interpretation of this fact is that the sun or other luminous body sends out rays of the same nature, distinguished only by their periodicity, which, by the action of a prism or other means, may be separated according to their wave lengths, and at any point of the spectrum there is but *one* ray and that it is *simple.*

344. All bodies absorb, transmit, and diffuse the radiations which they receive. Whatever part of the energy is received and not transmitted must perform work or store up energy in the molecule. If u represent the coefficient of transmission, t the thickness of the absorbing body, and I the original intensity of the luminous, heat or chemical effect of a radiation, then we have

$$I' = Iu^t \tag{404}$$

for the transmitted intensity I' of either of these effects considered; the ratio I' to I being a constant for each ray for the transmitted heat, light, or chemical effect. Experiment proves that the coefficient u is the same for each of these effects, when we consider the transmission of any particular radiation of wave length λ, and it varies in value for different values of λ. Thus, red glass is quite transparent for red rays, and it is equally transparent for the heat and chemical properties of the red ray; but it is not transparent for the yellow ray for either the luminosity, heat, or chemical effect of this ray.

345. The principle of absorption may then be stated to be, that when any radiation is transmitted through any medium, its three properties of luminosity, heat, and chemical effect are transmitted in equal proportion. When $u = 1$, the body is perfectly transparent for all effects; it is then said to be diathermous, and chemically and luminously transparent. Rock salt is an example of a body transparent for all radiations, and other bodies, as calcium

fluoride, quartz, and glass are less transparent for the ultra red rays, while alum, citric acid, ice, and water, stop these rays almost completely. Black bodies stop the luminous and ultra-violet, but let the ultra-red pass; a solution of iodine in carbon bisulphide affords a striking example of such a body. In some bodies, generally colorless and transparent, u diminishes as μ increases, while in transparent bodies of a blue or violet color, u increases with μ. In the same way, all bodies may be classed with respect to their powers of absorption.

346. *Emission.* When bodies are heated, they emit radiations, which however are not simple, but complex, being a mixture of rays of varying refrangibilities, determined by the temperature of the body. Thus, for Centigrade temperatures below 100°, the rays emitted are those which pass through rock-salt with difficulty; from 100° to 525°, the rays are still obscure; at 720°, the rays lie between A and C, and at 780° they extend to G, while at 1165° they include all to H. The emissive power of a body is directly proportional to its power of absorption, and this ratio is found to be a constant for all bodies; hence the ratio

$$\frac{E}{A} = C = 1,$$

expresses the law for all bodies, in which E is the emissive and A the absorptive power of any body for a particular radiation, and C is the emissive power of black bodies, whose absorption is taken equal to unity; both E and A depend on the temperature of the body and the refrangibility of the radiation. A striking analogy with respect to sound exemplifies this law. Thus, while the molecules of an incandescent gas can only emit radiations of a certain periodicity, experiment shows that they absorb radiations of the same periodicity. So in the case of a composite sound, a tuning-fork mounted on its resonant box will, by sympathetic resonance, enter into vibration when any component sound of equal wave length exists in the composite sound. The fork then absorbs this particular vibration, which is the only one it can emit, and thus becomes itself a new origin of sound, sending out in all directions waves of this single pitch; but if the tone corresponding to the pitch of the fork be wanting in the original composite sound, the

latter will not lose energy, but continue its way without sensible diminution.

347. If the homogeneous light of a sodium flame be deviated by a prism arranged with its lens to produce a pure spectrum, the latter will consist of a single *bright* line in the yellow, coincident with Frauenhofer's line D. If *lithium, thallium,* and *indium* be substituted in succession for sodium, their spectra will also be bright lines, but colored *red, green,* and *blue,* respectively ; and finally, if a mixture of these four metals be vaporized by combustion, the spectrum of their combined light will consist of these four bright lines, arranged in the order of their refrangibility. Other substances, when in a glowing state of vapor, are found to give one or more characteristic bright lines, having particular positions in the normal spectrum. These spectra are called *discontinuous.* When *solids* are raised by heat to incandescence, their spectra are *continuous,* and as their temperature increases and they become white-hot, their spectra extend from the red continuously to include the violet. Finally, the spectrum of solar light, whether examined directly or by reflection, and that of the fixed stars, are found to be crossed by numerous dark lines, in positions peculiar to each luminous source. *Spectrum analysis* in recent years has given a rational explanation of these phenomena, and has placed within our reach means of research which, at the same time, have largely increased our knowledge of the physics of the sun and stars, enlarged our powers of chemical analysis, and extended our list of the elements of matter. So great in extent is this new domain of physics, that a mere passing allusion to its more important principles is all that can be referred to here.

348. The instruments employed in this branch of physics are *spectroscopes* and *spectrometers.* They consist essentially of an accurate slit attached to a collimating telescope, through which the light to be examined is brought in parallel rays to incidence on a prism or train of prisms in their position of minimum deviation, and of an observing telescope armed with a micrometer eye-piece to view the rays after passing through the train ; the spectrometer has in addition an accurately divided circle. As these instruments will be constantly used in the lectures on this subject, an extended description is omitted.

349. Kirchoff found that when the brilliant light produced by incandescent lime passes through a *sodium* flame, the spectrum produced by it was continuous, except in the line D. When it traversed in succession the glowing vapors of *potassium, strontium,* etc., the bright lines which they would have formed alone were found wanting in the resulting spectra. These are called, for this reason, *absorption spectra.* This property of the luminous flames of substances in a state of vapor of absorbing the light which they emit, is found by experiment to belong to all bodies to which its application has been made, and the conclusion has therefore been drawn that the dark lines in the solar spectrum result from the absorption exercised by the luminous vapors of those substances existing in the solar atmosphere, which would produce bright lines of the same refrangibility in a state of glowing vapor. This inference, which has as a result that these dark lines would be *reversed* and become bright, could their spectra be seen without the influence of the direct sunlight, was first confirmed by the authentic observations of Professor Young during the solar eclipse of 1870.

350. It has been inferred also that gaseous bodies have the property of absorption at a lower temperature than that of incandescence, by the fact that many of the dark lines in the solar spectrum are due to the absorption exercised by aqueous vapor in the atmosphere. This fact has also been shown by direct experiment to belong to other gases; thus, when nitrous oxide or iodine vapor is interposed in the track of the solar beam, a number of dark lines are found to be present which do not occur in the normal spectrum. Then when this light is examined by the eye, its color would be modified by the loss of the concurrent influence of the absorbed rays. Liquid solutions and solids likewise absorb certain constituents of white light, and transmit others more freely, and the color of the transmitted light would result from the combined effect of those which are permitted to pass.

351. If we represent the original intensities of the elements of white light by the capital first letters of the names of their colors, and by the small letters the quantity represented by u for each color, we will have, from Eq. (404), as an expression for the intensity of the transmitted light,

$$Rr^t + Oo^t + Yy^t + Gg^t + Bb^t + Ii^t + Vv^t. \qquad (405)$$

A discussion of this expression will explain why the transmitted light may vary in color, intensity, or in hue, by varying the thickness.

352. The color which bodies possess is due then to the constituents of white light which are either transmitted or reflected from their surfaces. Certain constituents are absorbed at the surface, and if the remainder is reflected, the body is colored and opaque; if the remainder is partially transmitted and reflected, it will be transparent and colored due to the former, and the color by transmission may in some cases be different from that by reflected light.

353. The constants of color are *purity*, *intensity*, and *hue* or *tint*; the first has reference to the relative amount of white light mixed with it, the second depends on the light sensation arising from the energy of vibration, and the third is determined by its refrangibility, referred to the normal spectrum.

Mixed colors arise from adding the pure colors of the spectrum together. A new color which is not in the spectrum is *purple*, obtained by a mixture of the red and violet. The general results obtained from the mixture of colors may be briefly stated to be these, viz.: A mixture of two kinds of pure colored light produces a color in which the eye fails to detect the ingredients, in the same sense that the ear can analyze a compound tone; that the same mixed color can be produced in several ways; that many mixtures of two colors can produce *white*; in this case, the colors are said to be *complementary*; that a mixture of three or more colors gives no new hues, other than those which can be formed by a mixture of two colors; that all the hues between red and yellowish-green, and between blue-green and violet, will give hues located between the same limits, respectively, and that green when mixed with any other color gives no new hue, but rather a color less saturated than that with which it is mixed. The study of mixed colors is very much facilitated by the use of Maxwell's color-disks, by means of which two or more colors may be united in various proportions. The effects are somewhat modified from those produced by the pure colors of the spectrum, because of the impurity of the pigments employed, and from the effects of absorption by the material of the disk.

354. ***Phosphorescence and Fluorescence.*** Certain substances, such as the sulphur compounds of calcium, strontium, and barium, possess the property of emitting luminous rays in the dark for a shorter or longer time, provided they have been previously subjected to the action of light containing the more refrangible rays of the spectrum. This property is called *phosphorescence.* Other substances possess this property in a much less degree, such as uranium glass, quinine solutions, chlorophyll, æsculin solution, fluor-spar, and many others, in which the phenomenon lasts but for a very short time. This is called *fluorescence,* from the fact that it was first noticed in a species of fluor-spar. Both are essentially the same in principle, differing only in degree, and both are merely the effects of absorbed light. The radiations absorbed by these bodies are those of greater refrangibility than the rays they emit while exhibiting these properties, and the conclusion is that the absorbed energy of their ether molecules is given up to the molecules of the bodies, which in turn control the new vibrations of their surrounding ether, forcing them to take up vibrations of less periodicity. From this stored work, the expenditure of which is not instantaneous, the vibrations are continued until all is exhausted, when the phenomenon ceases. A new exposure to the more refrangible rays will again cause the same effects to be produced.

355. ***Achromatism.*** Since the refractive indices differ in the same medium for different colors, the results obtained in geometrical optics for the deviation of light by refraction are only true for homogeneous light. After refraction by a lens, a pencil of white light does not converge to, or diverge from a single point; first, because of the spherical aberration of the lens, and second, because of the unequal refrangibility of the colored rays of which the beam is composed. This latter departure from accurate convergence to a single point is called *chromatic aberration,* and is independent of the figure of the lens, but depends on the material of which it is made. Thus, if a lens L, Figure 85, intercept a beam of white light, the violet rays will first be brought to a focus at v, and the red rays will be brought to a focus at r; other colors will have their foci at intermediate points, in the order of their refrangibility.

If a screen be moved along the axis, the section of the deviated

cones between L and c will be red on the exterior and violet within, while beyond c the sections will be violet without and red within. The section ab at c, being the intersection of the cone of violet and red rays, is the least through which all the colored rays pass, and is therefore called the *circle of least chromatic aberration*.

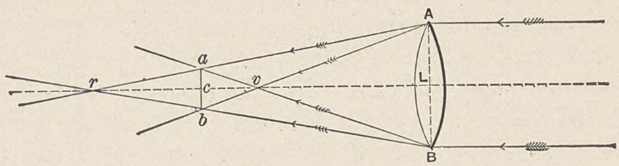

Figure 85.

Each pencil of rays proceeding from an object will likewise be separated into numerous cones of colored rays, and the object will have many colored images formed in different focal planes; when these images are viewed by the eye-lens, the resulting image will be confused, indistinct, and colored.

356. To reduce the dispersion of light in passing through lenses is the object of an *achromatic* combination. The possibility of such a combination arises from the fact that the *dispersion* of a ray of white light and its mean *deviation* are not proportional for different media. If dispersion were proportional to deviation by different media, then any combination which would destroy dispersion would also destroy deviation, and the combination would be as valueless for a telescopic object-lens as a piece of plate-glass. There are media, however, which produce the same dispersion, while the deviation differs. The achromatism is effected generally for two colors, preferably for those which are complementary. Owing to the irrationality of dispersion, the remaining colors are not accurately united, and the resultant effect is still somewhat colored, giving rise to a *secondary spectrum*, in which the fixed lines may be inverted, depending upon the relative dispersive powers of the two media.

357. Resuming Eqs. (295) and (296), Art. 266, and supposing the incidence small upon a thin prism, we have, taking the angles for the sines,

$$\phi = \mu\phi', \tag{406}$$

$$\phi_{,,} = \mu\phi_{,}\tag{407}$$

$$\alpha = \phi' + \phi_{,}, \tag{408}$$

$$\delta = \phi + \phi_{,,} - \alpha. \tag{409}$$

From the first two, we have

$$\phi + \phi_{,,} = \mu\,(\phi' + \phi_{,}) = \mu\alpha; \tag{410}$$

which, when substituted in the last, gives

$$\delta = (\mu - 1)\,\alpha. \tag{411}$$

Therefore, the deviation by a thin prism, with nearly normal incidence, is equal to the excess of the refractive index over unity, multiplied by the refracting angle of the prism.

The deviation by two such prisms is

$$\delta_2 = (\mu - 1)\,\alpha + (\mu' - 1)\,\alpha', \tag{412}$$

and for any number, $\delta_n = \Sigma\,(\mu - 1)\,\alpha.$ (413)

If the two prisms be turned so as to have their angles in opposite directions, we have

$$\delta_2 = (\mu - 1)\,\alpha - (\mu' - 1)\,\alpha'; \tag{414}$$

and if $\delta_2 = 0$, we must have as a condition,

$$\frac{\alpha}{\alpha'} = \frac{\mu' - 1}{\mu - 1}. \tag{415}$$

Considering μ and μ' to apply in succession to the violet ray or line H, and then to the red ray or line B, we have

$$\delta_v = (\mu_v - 1)\,\alpha - (\mu'_v - 1)\,\alpha', \tag{416}$$

$$\delta_r = (\mu_r - 1)\,\alpha - (\mu'_r - 1)\,\alpha'. \tag{417}$$

The condition that the difference of deviation should be zero for these rays is

$$\frac{\mu_v - \mu_r}{\mu'_v - \mu'_r} = \frac{\alpha'}{\alpha}. \tag{418}$$

Therefore the two prisms will be achromatized for red and violet when the ratio of the coefficients of dispersion of the media is inversely as the ratio of the refracting angles of the prisms.

358. The following table gives the values of the refractive indices from which these coefficients can be obtained, for water, crown and flint glass:

TABLE OF REFRACTIVE INDICES.

REFRACTING SUBSTANCES.	B. μ_r	C. μ_o	D. μ_y	E. μ_g	F. μ_b	G. μ_i	H. μ_v
Water..................	1.330935	1.331712	1.333577	1.335851	1.337818	1.341293	1.344177
Crown Glass, No. 13....	1.524312	1.525299	1.527982	1.531372	1.534337	1.539908	1.544684
Flint Glass, No. 13......	1.627749	1.629681	1.635036	1.642024	1.648260	1.660285	1.671062

359. ***Achromatism of Lenses.*** For a given radiant distance f, we have for the focal distance of red rays, by Eq. (314), Art. 276, deviated by two lenses.

$$\frac{1}{f_r''} = (\mu_r - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + (\mu'_r - 1)\left(\frac{1}{r''} - \frac{1}{r'''}\right) + \frac{1}{f}, \quad (419)$$

$$\frac{1}{f_v''} = (\mu_v - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + (\mu'_v - 1)\left(\frac{1}{r''} - \frac{1}{r'''}\right) + \frac{1}{f}; \quad (420)$$

whence, when $f_r'' = f_v''$, we have after reduction,

$$\frac{\mu_v - \mu_r}{\mu'_v - \mu'_r} = -\frac{\frac{1}{r''} - \frac{1}{r'''}}{\frac{1}{r} - \frac{1}{r'}}. \quad (421)$$

The first lens being supposed determined upon, r and r' are known, and r'' is usually taken to be equal to r', as the two lenses are generally in contact throughout; r''' is then the only unknown quantity in the above equation, and can readily be computed.

The first members of Eqs. (418) and (421) being the same, we have

$$\frac{\alpha'}{\alpha} = -\frac{\frac{1}{r''} - \frac{1}{r'''}}{\frac{1}{r} - \frac{1}{r'}}; \quad (422)$$

whence we see that the problems of achromatism for lenses become

those of prisms of the same material. Dividing both members of Eq. (421) by $\frac{\mu_g - 1}{\mu'_g - 1}$, we have

$$\frac{\frac{\mu_v - \mu_r}{\mu_g - 1}}{\frac{\mu'_v - \mu'_r}{\mu'_g - 1}} = -\frac{(\mu'_g - 1)\left(\frac{1}{r''} - \frac{1}{r'''}\right)}{(\mu_g - 1)\left(\frac{1}{r} - \frac{1}{r'}\right)} = -\frac{F_2}{F'_2}. \tag{423}$$

Therefore an achromatic combination of two lenses for red and violet can be formed, when the ratio of the dispersive powers of the media is negative and directly as the principal focal distances of the lenses. For an achromatic objective, the negative lens must have the greater power in order that a real focus may exist. The focal lengths of the lenses can readily be found from the preceding table for an achromatic combination composed of crown and flint glass lenses.

360. To find the chromatic aberration of a lens, and the diameter of the least chromatic circle, we have for the red and violet focal distances,

$$\frac{1}{f_r''} = (\mu_r - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f}; \tag{424}$$

$$\frac{1}{f_v''} = (\mu_v - 1)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f}. \tag{425}$$

Subtracting the first from the second, we have

$$\frac{1}{f_v''} - \frac{1}{f_r''} = \frac{f_r'' - f_v''}{f_v'' f_r''} = (\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{r'}\right). \tag{426}$$

Multiply the second member by $\frac{\mu_g - 1}{\mu_g - 1}$, and substitute in the first member $f_g''^2$ for $f_v'' f_r''$, and there results after substituting D and F_2 for the dispersive power and principal focal distance respectively,

$$f_r'' - f_v'' = \frac{D f_g''^2}{F_2}. \tag{427}$$

361. In Figure (85) we have approximately,

$$\frac{cv}{ab} = \frac{vL}{AB}, \qquad \frac{cr}{ab} = \frac{rL}{AB}; \tag{428}$$

whence
$$\frac{rv}{ab} = \frac{Lr + Lv}{AB} = \frac{f_g''}{AL} = \frac{f_g''}{a}; \tag{429}$$

in which f_g'' is taken to be the distance of the circle from the lens, and a is the semi-aperture; whence we have

$$ab = \frac{a \cdot rv}{f_g''} = \frac{aD \cdot f_g''}{F_2}; \tag{430}$$

hence the diameter of the circle of least chromatic aberration varies as the aperture and dispersive power directly. When the rays are parallel, f_g'' is very nearly equal to F_2, and the diameter ab becomes equal to aD.

362. By Eq. (423) we see that the conditions of achromatism depend only on the *focal* lengths of the component lenses, and are independent of the *forms* and *order* in which they are placed. The spherical aberration depending on the latter, it is therefore possible by a suitable arrangement of the forms and order to secure a combination which shall be both aplanatic and achromatic at the same time; such a combination is made in Huyghen's ocular.

363. ***Rainbow.*** The rainbow is a phenomenon frequently seen during a shower of rain when we turn our backs to the sun and view the opposite portion of the sky. It consists generally of two colored bands in each of which the succession of colors is the same as in the spectrum. The most brilliant, called the *primary* bow, has the *red* outside and the *violet* within; the less brilliant, called the *secondary*, is above the primary and has the *violet* without and the *red* within. The common centre of the bows is on the right line drawn from the sun through the eye of the observer. The space between the bows is darker than the rest of the sky, and in brilliant bows looks like a dark ribbon fringed with prismatic colors. Below the primary and above the secondary other bows, colored red and violet, are often seen, which are known as *supernumerary* or *spurious* bows.

364. The complete theory of the rainbow which has been given by Airy, is based upon the form of the emergent wave after refraction by the rain-drops, and is a consequence of the theory of caustics. The explanation ordinarily given of this phenomenon, that of Descartes, is inexact and merely approximate, fails to

account for the supernumerary bows, and is not confirmed by exact measurement.

Let the spherical drop of water I, I′, I″, Fig. 86, intercept the parallel rays of homogeneous light coming in the direction SI, and let i and r be the angle of incidence and refraction. The ray SI will be refracted to I′, reflected to I″, and emerge in the direction I″R. Let δ be the deviation of any ray from its direction before incidence to its direction after emergence. Then the deviation produced by the first refraction at I is $i - r$, by the first reflection at I′ is $\pi - 2r$, and by the second refraction at I″ is $i - r$, since the angle of incidence is here r and angle of refraction is i. Then the total deviation is

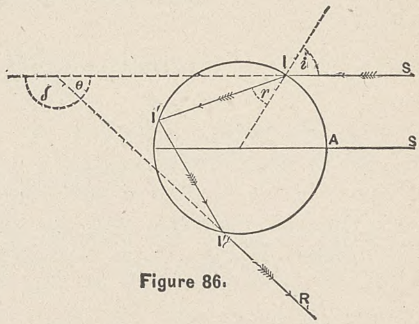

Figure 86.

$$\delta = 2(i - r) + (\pi - 2r), \tag{431}$$

and for any number of internal reflections will be

$$\delta = 2(i - r) + n(\pi - 2r). \tag{432}$$

To find the value of i corresponding to a maximum or minimum deviation, differentiate Eq. (432) and place $\frac{d\delta}{di}$ equal to zero, and we have

$$1 - (n + 1)\frac{dr}{di} = 0. \tag{433}$$

From
$$\sin i = \mu \sin r, \tag{434}$$

we have
$$\frac{dr}{di} = \frac{\cos i}{\mu \cos r}, \tag{435}$$

which substituted in Eq. (433) gives

$$1 - (n + 1)\frac{\cos i}{\mu \cos r} = 0; \tag{436}$$

whence
$$\mu^2 \cos^2 r = (n + 1)^2 \cos^2 i. \tag{437}$$

Eliminating $\cos^2 r$ by Eq. (434) we have

$$\mu^2 = 1 + (n^2 + 2n)\cos^2 i; \tag{438}$$

whence

$$\cos i = \sqrt{\frac{\mu^2 - 1}{n^2 + 2n}}. \tag{439}$$

365. For water μ is about $\frac{4}{3}$, and hence $\mu^2 - 1$ is $\frac{7}{9}$; $n^2 + 2n$ is at least 3; therefore the value of i is always admissible, and whatever may be the number of internal reflections, there always exists for a given color a single value for the angle of incidence for which the deviation is a maximum or minimum. Rays which have this angle of incidence or differ but little from it, are called when they emerge *efficacious* rays. To find whether this value of i corresponds to a maximum or minimum deviation, we have

$$\frac{d^2\delta}{di^2} = -2(n + 1)\frac{d^2 r}{di^2}, \tag{440}$$

from which we see that the sign of $\frac{d^2\delta}{di^2}$ is always contrary to that of $\frac{d^2 r}{di^2}$.

From the relation,

$$\frac{dr}{di} = \frac{\cos i}{\mu \cos r}, \tag{441}$$

we have

$$\frac{d^2 r}{di^2} = \frac{-\mu^2\cos^2 r \sin i + \mu \sin r \cos^2 i}{\mu^3 \cos^3 r} = \frac{(1 - \mu^2)\sin i}{\mu^3\cos^3 r}; \tag{442}$$

whence we see that $\frac{d^2 r}{di^2}$ is always negative and hence $\frac{d^2\delta}{di^2}$ is always positive. Therefore the deviation of the efficacious rays is always a minimum, whatever be the number of reflections.

366. To determine the position of the point of emergence of any ray SI with reference to the point of emergence of the normal ray SA, we have $(n + 1)\pi$ for the value of the angle between A and the point of emergence of SA after any number of reflections; and for the corresponding angle between the point of emergence of SI and I the expression $(n + 1)(\pi - 2r)$; representing the required angle by θ, we have

$$\theta = 2(n+1)r - i. \tag{443}$$

Applying the test for maxima and minima we have

$$\frac{d\theta}{di} = 2(n+1)\frac{dr}{di} - 1 = 0; \tag{444}$$

whence by reduction

$$\cos^2 i = \frac{\mu^2 - 1}{4(n+1)^2 - 1}. \tag{445}$$

The value of cos i for one reflection is $\sqrt{\frac{7}{135}}$, a very small quantity which diminishes very rapidly as n increases; besides as we have

$$\frac{d^2\theta}{di^2} = 2(n+1)\frac{d^2r}{di^2}, \tag{446}$$

and since $\frac{d^2r}{di^2}$ is always negative, $\frac{d^2\theta}{di^2}$ is negative and corresponds to a maximum for the value of cos i above.

367. The angular distance θ of the point of emergence of the ray SI to the point of emergence of the normal ray SA is zero when SI coincides with SA, then increases with an increase in i, and attains a certain maximum when the angle of incidence is near 90° ($i = \cos^{-1}\sqrt{\frac{7}{135}}$), that is when the incident ray is nearly tangent to the spherical drop.

368. The primary bow corresponds to one and the secondary to two internal reflections. Assuming from the table the values of μ 1.331 and 1.344 for the extreme red and violet rays for water, we have for the primary bow,

for *red* $i = 59° 32'$, $r = 40° 21'$, $\delta = 137° 40'$.

Angular radius of red $= \pi - \delta = 42° 20'$;

for *violet* $i = 58° 44'$, $r = 39° 30'$, $\delta = 139° 28'$.

Angular radius of violet $= \pi - \delta = 40° 32'$;

and therefore the angular breadth of the bow for the rays proceeding from one point of the solar disk,

$$42° 20' - 40° 32' = 1° 48'.$$

Considering the pencils proceeding from all points of the solar disk, this breadth would be increased by the sun's apparent diameter, or about 32′, and therefore the whole breadth of the primary bow would be about 2° 20′.

For the secondary bow we would have in the same way,

for *red* $i = 71° \ 54'$, $r = 45° \ 34'$, $\delta = 230° \ 24'$.

$$\text{Angular radius of the red} = \delta - \pi = 50° \ 24' ;$$

for *violet* $i = 71° \ 29'$, $r = 44° \ 52'$, $\delta = 233° \ 46'$.

$$\text{Angular radius of the violet} = \delta - \pi = 53° \ 22' ;$$

and for total breadth, $53° \ 22' - 50° \ 24' + 32' = 3° \ 30'$.

369. Now if the direction of any efficacious ray due to one internal reflection I″R, Fig. 86, be produced backward, and be supposed revolved about the axis of the bow, the arc marked out by its extremity will be the arc of the bow for that color. All raindrops which meet the conical surface described by the revolution of this ray, contribute in passing, their few rays of that particular color which reach the eye of the observer, supposed at the vertex of the cone. And since the semi-angle of the cone is 42° 20′ for red, and varies by continuity for the other colors of the spectrum to 40° 32′ for violet, therefore in the primary bow the red is without and the violet within. When the efficacious ray I‴R, Fig. 87, due to two internal reflections is in like manner revolved, the semi-angle at the vertex for violet is 53° 22′, and for red is 50° 24′, and therefore the violet is without and the red within in the secondary bow.

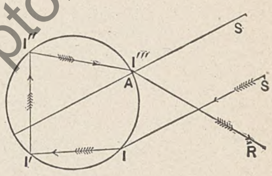

Figure 87.

All rays either once or twice internally reflected while the drops are passing within the bows, pass either above or below the eye, and hence, this space sending fewer rays, to the observer appears less bright than the rest of the sky.

370. This explanation of the rainbow, with the values of its dimensions, is based on the supposition that the positions of the emergent efficacious rays correspond to the only maximum. Let us construct, as in Fig. 88, the paths of rays incident above the

central ray, which undergo one internal reflection. Those that are incident between A and I are first reflected between A′ and I′, and emerge between A and I″; those incident between I and the tangent ray near T are reflected between I′ and a point near A′, and emerge between B and I″. The consecutive intersections of the emergent rays therefore form a caustic of two branches, one A′C′ normal to the spherical drop at A′, and the other BC tangent to the drop at B, very near I″, as shown in the figure. The efficacious ray after emergence I″E is an asymptote to these caustics. The emergent wave is the development of the caustic surface formed by revolving the caustic curves about the ray SA, which is undeviated.

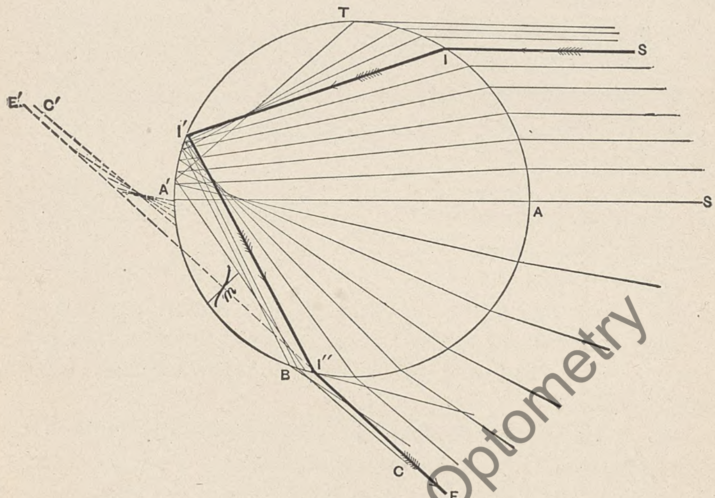

Figure 88.

A section of the emergent wave by the plane of the figure is a curve having a point of inflection at m on the efficacious ray. The intensity of the luminous effect in the neighborhood of a caustic has been thoroughly discussed by Airy, and the numerical values of his integrals, when applied to the theory of the rainbow, are in perfect accord with the observed phenomena. It results that for positions near, but to the right of the efficacious ray, as shown in the figure, there are series of maxima and minima luminous intensity, of which the first maximum not exactly coincident with I″E corresponds to the primary bow; the other maxima, still far-

ther from I″E, belong to the supernumerary bows. The deviation of the primary bow is less than that determined by the Cartesian theory above, and is found to decrease slightly with the decrease in the diameter of the rain-drops. The same modifications occur in the secondary, and in its supernumerary bows, located above it.

371. ***Interference of Light.*** The principles enunciated in Part I, Arts. 64 to 72, being applicable to all undulatory motion, will therefore be assumed to govern in the succeeding discussion. To ascertain the circumstances under which two luminous waves will interfere, let S, in Figure 89, be a luminous source from which *homogeneous* rays of wave length λ proceed; M and M′ two plane mirrors which intercept and deviate the rays, and whose inclination is indicated by i, as in the figure. Then, by construction, s and s' are the virtual foci of S, and the path of the rays after deviation by the mirrors will be the same as if, starting in the same phase, they had proceeded from two separate luminous origins s and s'.

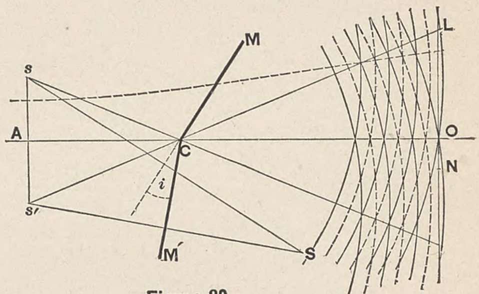

Figure 89

Join A the centre of ss' with C by a right line, and extend it indefinitely to O; let N be any near point to O on LO, drawn at right angles to CO, and let $sC = a$ and $CO = b$. Since i is the inclination of the mirrors, the angle sCs' is $2i$.

372. To find the resultant displacement of a molecule at N, due to the two undulations proceeding from s and s', we will assume the general expression for the displacement, Eq. (69),

$$\delta = \alpha \sin\left[\frac{2\pi}{\lambda}(Vt - x) + A'\right], \tag{447}$$

in which δ is the displacement at any time t, α the maximum displacement, λ the wave length, V the velocity of propagation, x the distance of the molecule from the source, and A' the arbitrary arc

which determines the phase. If c is the value of the maximum displacement at a unit's distance from the origin, we have, by Art. 72,

$$\alpha = \frac{c}{x}.$$

Substituting this value for α, and $\frac{2\pi}{\lambda} A$ for A', the displacements of N due to the waves from s and s' will be, respectively,

$$\left.\begin{aligned} \delta &= \frac{c}{s\mathrm{N}} \sin\left[\frac{2\pi}{\lambda}(Vt - s\mathrm{N} + A)\right], \\ \delta' &= \frac{c}{s'\mathrm{N}} \sin\left[\frac{2\pi}{\lambda}(Vt - s'\mathrm{N} + A)\right]; \end{aligned}\right\} \tag{448}$$

and, since AO may replace sN or s'N in the coefficient without appreciable error, the resultant displacement, by Art. 65, at N will be

$$\frac{c}{\mathrm{AO}}\left\{\sin\left[\frac{2\pi}{\lambda}(Vt - s\mathrm{N} + A)\right] + \sin\left[\frac{2\pi}{\lambda}(Vt - s'\mathrm{N} + A)\right]\right\}, \tag{449}$$

which may be written,

$$\frac{2c}{\mathrm{AO}} \cos\left[\frac{\pi}{\lambda}(s\mathrm{N} - s'\mathrm{N})\right] \sin\left[\frac{2\pi}{\lambda}\left(Vt - \frac{s\mathrm{N} + s'\mathrm{N}}{2} + A\right)\right]. \tag{450}$$

The maximum resultant displacement is therefore

$$\frac{2c}{\mathrm{AO}} \cos\left[\frac{\pi}{\lambda}(s\mathrm{N} - s'\mathrm{N})\right], \tag{451}$$

and the maximum intensity of the light is

$$\frac{4c^2}{\overline{\mathrm{AO}}^2} \cos^2\left[\frac{\pi}{\lambda}(s\mathrm{N} - s'\mathrm{N})\right]. \tag{452}$$

From the figure we have

$$\left.\begin{aligned} \overline{s\mathrm{N}}^2 &= \overline{\mathrm{AO}}^2 + (s\mathrm{A} + \mathrm{ON})^2 \\ &= (a \cos i + b)^2 + (a \sin i + \mathrm{ON})^2. \end{aligned}\right\} \tag{453}$$

$$\left.\begin{aligned} \therefore\ sN &= a\cos i + b + \frac{1}{2}\cdot\frac{(a\sin i + ON)^2}{a\cos i + b},\ \text{very nearly;} \\ s'N &= a\cos i + b + \frac{1}{2}\cdot\frac{(a\sin i - ON)^2}{a\cos i + b},\ \text{very nearly;} \end{aligned}\right\} \quad (454)$$

$$\left.\begin{aligned} sN - s'N &= \frac{2a\sin i}{a\cos i + b}\,ON \\ &= \frac{2a\sin i}{a + b}\,ON,\ \text{very nearly, since } i \text{ is very small;} \end{aligned}\right\} \quad (455)$$

and the intensity at N will be therefore

$$\frac{4c^2}{(a+b)^2}\cos^2\left(\frac{2\pi}{\lambda}\cdot\frac{a\sin i}{a+b}\,ON\right), \tag{456}$$

which varies for all points N with the distance of N from O. Making ON in succession equal to

$$0, \quad \pm\frac{a+b}{a\sin i}\cdot\frac{\lambda}{4}, \quad \pm\frac{a+b}{a\sin i}\cdot\frac{2\lambda}{4}, \quad \pm\frac{a+b}{a\sin i}\cdot\frac{3\lambda}{4}, \quad \text{etc.}$$

We will have, for the corresponding intensities at the points N,

$$\frac{4c^2}{(a+b)^2}, \quad 0, \quad \frac{4c^2}{(a+b)^2}, \quad 0, \quad \text{etc.;}$$

and generally, for points for which

$$ON = \pm\frac{a+b}{a\sin i}\cdot\frac{2n\lambda}{4},$$

n being an integer, the intensity will be $\frac{4c^2}{(a+b)^2}$; and for points for which

$$ON = \pm\frac{a+b}{a\sin i}\cdot\frac{(2n+1)\lambda}{4},$$

the intensity will be zero, or there will be darkness. Intermediate values of ON will correspond to points at which all degrees of brightness will exist, from darkness to the maximum illumination. The same will be true for points in the plane OL, normal to OC, and there will then be a series of lines of alternate darkness and brightness, which are called *interference fringes*.

373. If from s and s', as centres, arcs of circles be described whose consecutive radii differ by $\frac{\lambda}{2}$, we readily see, since the waves start in the same phase, that at points where the difference of route is any even multiple of $\frac{\lambda}{2}$, the component displacements meet the molecule in the same phase and add their effects, while at points where this difference is an odd multiple of $\frac{\lambda}{2}$, the displacements meet in opposite phases and cause mutual destruction. The points of maximum brightness are indicated in the figure by the intersections of two full or two dotted curves, and the points of darkness by the intersection of a full and a dotted line. The locus of each of these series of points is an hyperbola, whose foci are s and s', since the difference of distance of any point of the locus from s and s' is equal to $\frac{n\lambda}{2}$, n being odd for the dark lines and even for the bright lines.

374. If instead of homogeneous light, white light is employed, the fringes corresponding to each constituent will be in sequence nearer to OC, according as the values of λ are smaller. Therefore the violet fringe will be the nearest, the blue next, and so on in succession to the red. At O all will be united, and their combined effect will be white. The position of the first fringe for any constituent, as violet, is determined by the value of

$$\text{ON} = \frac{a+b}{a \sin i} \cdot \frac{\lambda'}{2};$$

and its intensity would be measured by

$$\frac{4c^2}{(a+b)^2}.$$

At the same point the intensity for the blue would be

$$\frac{4c^2}{(a+b)^2} \cos^2 \pi \frac{\lambda'}{\lambda''};$$

and hence we see that the dark spaces would soon be encroached upon by the various colors, due to their difference of wave length, and therefore the number of distinct fringes in white light is

limited. The wave length λ is so small as to be incapable of direct measurement, or even appreciation by the eye, but the value of the expression

$$\frac{a+b}{a \sin i} \cdot \frac{\lambda}{2}$$

can be made large by making i small, and since a, b, and i can be measured directly, the value of λ can be determined for each constituent of white light.

375. This crucial experiment is one of fundamental importance in the undulatory theory of light. If light be not the result of undulatory motion, no rational explanation can be given of the fact that two lights added together do produce darkness, and that this result is due to the concurrent effect of both is satisfactorily shown by the fact that if one of the mirrors be covered or removed, the fringes at once disappear, and uniform illumination results.

An apparent difficulty arises from the fact that this property of interference does not occur when the two sources of light are independent and distinct; but this is satisfactorily accounted for by considering the impossibility of any two independent luminous sources sending out waves with exactly the same phase for even the smallest appreciable time. We may conceive this to happen for a few or even several thousand vibrations, but the probability that this should continue for millions of millions of vibrations, when the variations that affect the luminous sources are entirely independent of each other, is beyond conception. Now by many repeated observations, the mean wave length λ is found to be a little greater than 0.0000005 m., and since the velocity of light in air is about 300,000,000 m., it follows that the number of waves sent out from a luminous source of homogeneous light having this periodicity is about 600,000,000,000,000 in each second of time. Hence, in order that the waves from these two sources should interfere and for a second of time permit the fringe to be visible, this enormous number of waves must be sent out from each, without any variation as to phase, no matter what may be the variation in the causes that serve to keep the sources active during this period.

376. To explain this analytically, let δ and δ' represent the molecular displacements due to two waves proceeding from different sources, x and x' the distances of m from the sources, and ϕ and

ϕ' the phases of the vibration at any time t from the epoch; then we have for the displacements of the molecule due to each wave,

$$\delta = \alpha \sin 2\pi \left(\frac{t}{\tau} - \frac{x}{\lambda} + \phi\right),$$

$$\delta' = \alpha' \sin 2\pi \left(\frac{t}{\tau} - \frac{x'}{\lambda} + \phi'\right);$$

whence the resultant energy would be measured by

$$\alpha^2 + \alpha'^2 + 2\alpha\alpha' \cos 2\pi \left(\frac{x - x'}{\lambda} + \phi' - \phi\right).$$

The value of this expression depends as much upon $\phi' - \phi$ as upon $\frac{x - x'}{\lambda}$, and therefore as the difference of phase $\phi' - \phi$ of the concurrent waves will be affected by all the changes that are constantly affecting their sources, we see the absolute necessity of employing a single source in exhibiting the interference fringes. Otherwise, the expression $\phi' - \phi$ will have in general an indefinitely great number of different values, and the expression into which it enters will likewise have in an indefinitely short time corresponding values, which will be limited on the one side by its maximum $(\alpha + \alpha')^2$, and by its minimum $(\alpha - \alpha')^2$ on the other. Therefore the intensity of the light due to their concurrent effect, as estimated either by photometric measurement or by the eye, will be represented in any appreciable time by the mean of these values, or $\alpha^2 + \alpha'^2$, that is, by the sum of the separate intensities. When, however, the source is single and interference occurs, the intensity is either the square of the sum of the two maxima $(\alpha + \alpha')^2$, or of their difference $(\alpha - \alpha')^2$, depending on the point considered, and if as is usually the case, $\alpha = \alpha'$, these become respectively $4a^2$ and zero; that is, an intensity four times greater than that due to either or none whatever.

377. Instead of the mirrors, a double prism or two prisms united at their bases, having very small refracting angles, can be used to exhibit the interference fringes. In this, as in the previous case, the source is usually a line of light formed at the focus of a cylindrical lens, from which rays fall nearly normal to the back of the prism, and after deviation proceed as if they had

originated at the two virtual foci of the biprism. The distance of the foci s and s' from A in this case is equal to that found for the mirrors multiplied by $(\mu - 1)$, μ being the index of refraction of the prism. Hence, all other things being unchanged, the intensity and distance apart of the fringes are represented by

$$\frac{4c^2}{(a+b)^2}\cos^2\left[\frac{2\pi}{\lambda}\cdot\frac{a \sin i}{a+b}(\mu - 1)\,ON\right], \tag{457}$$

and $$\frac{a+b}{a \sin i}\cdot\frac{1}{4}\frac{\lambda}{(\mu - 1)} \text{ respectively.} \tag{458}$$

The breadth of the fringes for different colors will therefore depend on $(\mu - 1)$ as well as on λ, and since μ varies with λ, these fringes for different colors will be more unequal in breadth than in the case of the mirrors; consequently the colors by overlapping will lose distinctness at a short distance from O. The few, however, which enclose the bright white line at O are much more intense and distinct than those of the mirrors, because of the greater intensity of the light transmitted by the prism.

378. If in either of these cases a piece of thin glass or other transparent solid or liquid medium be interposed, so as to make the route of one of the interfering beams optically longer than the other, the fringes will be found displaced towards the corresponding side. By this means the velocity of light in air has been shown to be unquestionably greater than that in glass, and to be in the proportion of $\mu : 1$.

379. ***Colors of Thin Plates.*** The colors displayed in soap bubbles, thin films of coal tar on the surface of water, fractures in glass or in crystals of various kinds, and in other like phenomena, are due only to the principle of interference. If a lens of small curvature be placed on a piece of plane glass, the thin film of air of varying thickness between the two surfaces will cause incident light to interfere and produce colored rings of varying diameter. When homogeneous light is used the rings are alternately dark and bright when seen by reflected light, and bright and dark by transmitted light. With incident white light the rings are variously colored and form a perfectly determinate and easily reproduced series, so that they can be used as a reliable

standard for color. They are known as *Newton's rings,* and the succession of colors is called *Newton's scale.*

380. It is easily seen that the thickness of the plate of air increases from zero at the point of contact of the lens with the plate, and is always the ver-sine of the arc; it therefore varies as the square of the diameter of the ring. By accurate measurement Newton found that the squares of the diameters of the rings form an arithmetical progression, and hence the thicknesses of the thin plates for rings of the same color are in the same ratio. To explain in an elementary way their formation, let SI, S′I′, etc., be incident rays of homogeneous light, and suppose the two rays that move along I′R′ to be in a condition to interfere; part of S′I′ being directly reflected at I′, and part of SI being refracted and reflected along II″I′R′. To find the difference of route so as to determine the change of phase, draw the normal I′K. It is evident that the rays moving from I′ along I′R′ will be simultaneously in the same phase as that moving along IR at K, since these points are on the same plane normal to the reflected wave front; and since the velocities in air and glass are as $\mu : 1$, the distance KI reduced to its equivalent route in air is μKI. Again, from mechanical principles and from the analogy of reflected waves of sound in air, Arts. 228 and 229, we see that a difference of phase of 180°, or its equivalent $\frac{\lambda}{2}$ in route, must be added or subtracted from each reflection of a wave incident on a surface separating a rarer from a denser medium, while no difference is to be allowed, when reflection takes place on a surface separating a denser from a rarer medium. We therefore have for the difference of route for the two interfering rays,

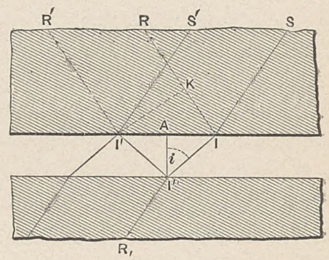

Figure 90.

$$\mathrm{II}'' + \mathrm{I}''\mathrm{I}' + \frac{\lambda}{2} - \mu\mathrm{KI} = 2n\frac{\lambda}{2}, \quad \text{or} \quad = (2n+1)\frac{\lambda}{2}, \tag{459}$$

for the maximum and minimum intensity respectively, in which n represents any integer marking the number of the interference.

Representing the thickness of the film of air by t, the angle of incidence, AI″I by i, and the angle of refraction $\frac{1}{2}$S′I′R′ by r, and since I′I″ = I″I, we have

$$2\mathrm{I'I''} - \mu\mathrm{KI} = (2n-1)\frac{\lambda}{2}, \quad \text{or} \quad 2n\frac{\lambda}{2}. \qquad (460)$$

But we have from the figure $\mathrm{II''} = \dfrac{t}{\cos i}$, and

$$\left.\begin{aligned} \mathrm{KI} &= \mathrm{I'I} \cos \mathrm{KII'} = 2\mathrm{AI} \sin r \\ &= 2\mathrm{II''} \sin i \sin r = 2t \tan i \sin r \end{aligned}\right\}. \qquad (461)$$

Whence after reduction we have

$$t = \frac{\lambda}{4}(2n-1) \sec i, \quad \text{or} \quad = \frac{\lambda}{4} 2n \sec i. \qquad (462)$$

Whence we see that the thickness corresponding to the bright rings are as the odd numbers 1, 3, 5, 7, etc., and to the dark rings are as the even numbers 0, 2, 4, 6, etc.; that the thickness for a ring of any given order varies with λ, and hence varies with the color and with the nature of the substance forming the film; and finally, that the thickness varies with the incidence i.

381. The transmitted rays R_1, etc., are also capable of interfering, and a similar discussion would show that the interference rings formed by refraction are complementary in color to the reflected rings, a deduction which is confirmed by experiment.

382. ***Diffraction.*** In the cases of interference above referred to, since the routes of travel of the interfering rays must differ by some odd multiple of $\frac{\lambda}{2}$, either reflection or refraction must take place upon a portion at least of the rays which setting out at the same phase proceed from a *single* origin. The principle of Huyghens is applicable, however, in all cases belonging to diffraction. By this principle the disturbance at P′, Figure 10, page 50, may be considered as the resultant of all the disturbances sent to it from every point of the wave front in any one of its anterior positions as DAC. Assume a series of distances beginning with P′A and successively increasing with the constant value $\frac{\lambda}{2}$; with these distances as elements conceive conical surfaces described

having P′ for their common vertex and P′A for their common axis. The zones which they intercept on the spherical wave front are annular surfaces slightly increasing in area as we proceed outward from A. From these *zones of Huyghens*, as they are called, secondary waves proceed which are all in the same phase at starting, but which, owing to the difference of route, meet at P′ in various phases. The secondary waves due to each zone vary by insensible degrees from phases of complete accord to phases bordering on complete discordance. The displacements of P′ due to waves from any two contiguous zones are therefore of opposite signs, and were the number of these waves exactly equal, their rays parallel, and the displacements of the same amplitude, mutual destruction of each other's effects on P′ would ensue. But while this equivalency is not exact, it is approximately so with respect to adjacent zones at a very short distance from A, and the intensity of the light at P′ will be represented by the decreasing converging series,

$$\mathrm{I} - \mathrm{I}' + \mathrm{I}'' - \mathrm{I}''' + \text{etc.}, \tag{463}$$

in which the terms depend on the aggregate displacements of the molecule at P′ due to each zone. Hence the resultant intensity of light at the point P′ is sensibly due to the concurrent effect of but a few of the zones, and whatever be the number considered, this intensity has always a value between I and I − I′. Therefore the maximum effect is that due to the central zone, and since λ is exceedingly small, we may conclude that light is propagated sensibly in right lines from its source.

383. The consequences of the above reasoning, which have been repeatedly confirmed by experiment, are

1°. That if by any means the effect of the even numbered zones 2, 4, 6, 8, etc., be stopped out as by opaque screens, the intensity of the light at P′ due to the zones 1, 3, 5, 7, etc., is very much increased. By a carefully contrived experiment, the light of the second zone has been obstructed and the intensity of the light which remained was found to be nearly five times as great as before.

2°. If a small circular opaque disk be interposed in the cone of rays proceeding from O, the intensity of the light at the central point of the geometric shadow is the same as when the disk is removed.

3°. A small circular aperture being interposed in the cone of rays, and properly placed with respect to the screen, will exhibit a perfectly dark spot at the projected centre of the aperture.

The analytical proof of these consequences, as well as of many others, all of which are easily confirmed by experiment, is beyond our assigned limits and is therefore omitted.

384. We have seen in Art. 251 that the extent of the luminous source has a marked influence in softening the shadows cast by opaque bodies. When the source is a mere point or a right line, the contour of the geometric shadow is not the true boundary between the light and darkness, but there are certain luminous effects both within and without the geometric shadow, which are perfectly explained by the principles upon which this subject of diffraction is founded. Let the parallel beam of homogeneous light S be brought to a focus at O by a lens of short focus, and suppose an opaque screen PM be placed in the diverging cone with its edge P in the axis OQ. The geometric shadow of this edge on QR is at Q. Below this line Q a faint illumination exists within

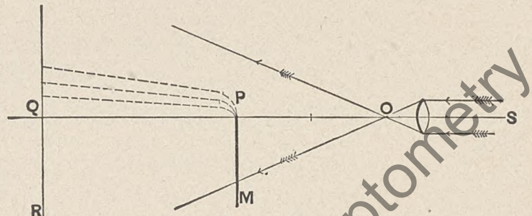

Figure 91

the shadow, which fades away gradually from Q. Above Q and parallel to it a series of alternately bright and dark bands or fringes are found whose distances apart vary with the position of the screen and with λ. Careful measurements show that their loci are hyperbolas whose vertices are at P and centres midway between O and P. If the incident light be white, the fringes are iris colored, the violet of each fringe being nearest the geometric shadow and red the farthest.

385. If the obstacle be very narrow, such as a fine wire or horsehair, the whole shadow is occupied by a series of very fine alternately bright and dark lines parallel to the wire; these lines

becoming wider as the diameter of the wire is diminished. If a rectangular aperture be formed whose width is small compared to its length, the diffraction fringes will be found in the luminous portion, which will become wider as the aperture becomes narrower. A series of rectangular slits, separated by opaque intervals, may be formed by ruling on glass a number of equidistant lines, by means of a diamond point. Such an apparatus is called a *diffraction grating*. When a beam of light falls upon it a direct image of the luminous surface is formed by reflection or transmission, flanked by a series of spectra each of which is separated from the preceding by a space devoid of light.

386. To explain the formation of the diffraction spectra by means of a grating, let the diaphragm MN be placed normal to the parallel beam of homogeneous light SS′, and suppose that *bac* be a rectangular opening whose width is very small compared to its length. Considering alone the effects on a screen PP′ in the plane MP′, we see that by the principle of Huyghens all the points of *bc* may be considered as centres of secondary waves from which rays proceed in all directions beyond MN. By the preceding prin-

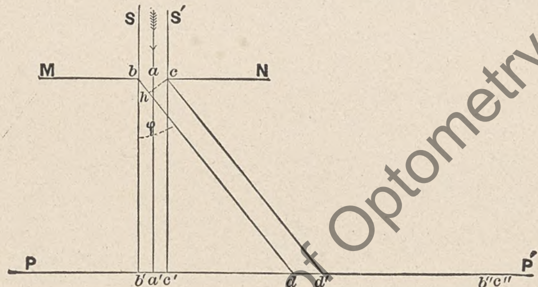

Figure 92

ciples the maximum disturbance will occur at a', and provided the difference of route $bc' - cc'$ be less than $\frac{\lambda}{2}$, the space $b'c'$ will have less illumination, but with inappreciable difference of intensity on either side of a'. Now consider the effects at positions on the right of the axial line aa', since whatever is proved true for these will be equally true for corresponding points on the left, and we will find a space dd' such that points within it have no disturbance due to

rays $bd \ldots cd'$ whose angle of diffraction ϕ is such that bh is equal to λ. For, beginning with a ray from b, we have a ray from a whose difference of route is exactly $\frac{\lambda}{2}$, and for each consecutive point in ba we have a corresponding point in ac, such that rays from each of these pairs neutralize each other within the space dd'. Still farther to the right a space $b''c''$ will be found such that the difference bh will be equal to $\frac{3}{2}\lambda$, and if we consider the centres in bac separated into three equal groups, the rays from the first group will completely neutralize those of the second, and therefore permit those of the third to produce concurrent effects in $b''c''$; hence $b''c''$ will be illuminated with an intensity much less, however, than that of $b'c'$; and so on for other alternate dark and bright spaces.

387. If now a bright line of light be used from which parallel rays proceed and a grating be interposed in their paths, the open spaces will permit the light to pass while the furrows will intercept it.

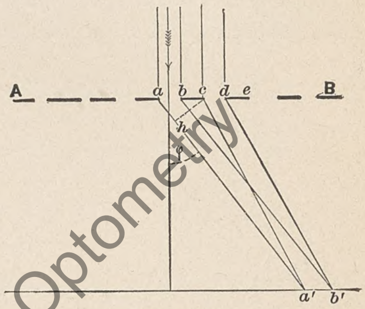

Figure 93

Let AB be such a grating, ab, cd, etc., the clear spaces, bc, de, etc., the furrows, and from what precedes it is evident that if ah be equal to $n\lambda$, that the rays at $a'b'$ will unite their effects and produce a fringe of the n^{th} order of wave length λ. If ah, however, differ by any small fraction of λ, as $\frac{1}{m}$, then the beam passing through the $\frac{m^{th}}{2}$ space from ab will be in discordance with that passing through ab, that passing through the $\left(\frac{m}{2}+1\right)$ space with that through the 2d, and so on. Hence the angle ϕ is determinate for the rays of a particular color. Therefore to determine λ for any fringe as the n^{th}, representing the width of a space and a furrow $(ab + bc)$ by ω, we have

$$\sin \phi_n = \frac{n\lambda}{\omega}. \tag{464}$$

388. If white light be employed, since λ varies for each constituent, the violet fringes will be found nearest the central line and the others arranged in the order of their wave lengths. The finely constructed gratings of Nobert, Rutherfurd, and others, give such pure spectra that the fixed lines of Frauenhofer are clearly defined, and thus afford means of measurement for the wave lengths corresponding to their refrangibility. Knowing the number of ruled lines to the inch, the value of ω is known; the angle ϕ_n is measured by the spectrometer and therefore λ results from the solution of the above equation. The advantage of this method of determining λ consists in the fact that the distances of the fixed lines in the diffracted spectra are always proportional whatever be the substance of the grating, while in the prismatic spectrum they vary with the dispersive power of the medium of the prism. Thus since $\sin \phi$ is very small, it may be replaced by $\phi \sin 1'$, and if ϕ_1, ϕ_2, ϕ_3, represent the deviations corresponding to any three points of the n^{th} spectrum, and λ_1, λ_2, λ_3, the corresponding wave lengths, we have

$$\phi_1 = \frac{n\lambda_1}{\omega \sin 1'}, \qquad \phi_2 = \frac{n\lambda_2}{\omega \sin 1'}, \qquad \phi_3 = \frac{n\lambda_3}{\omega \sin 1'}, \tag{465}$$

whence

$$\frac{\phi_2 - \phi_1}{\phi_3 - \phi_1} = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}; \tag{466}$$

but $\phi_2 - \phi_1$ and $\phi_3 - \phi_1$ are the distances of the lines λ_2 and λ_3 from λ_1 in the n^{th} spectrum, and are by Eq. (466) directly proportional to the increase in wave length of λ_2 and λ_3 over that of λ_1. Since the latter are constant for given lines of the spectrum, the ratio of the increase of $\phi_2 - \phi_1$ to $\phi_3 - \phi_1$ must be constant. We also see that the consecutive diffraction spectra increase in length from the centre and will soon overlap, so that the red of the n^{th} will fall on the violet of the $(n+1)^{th}$.

389. By the above method the values of the wave lengths have been determined with the greatest accuracy by many observers. The agreements in measurement extend to the units of the millionths of the millimetre; those adopted in the following table are from the investigations of Van der Willigen in 1868, as given in the Archives of the Musée Teyler.

WAVE LENGTHS OF THE FIXED LINES IN THE NORMAL SPECTRUM.

FIXED LINES.	WAVE LENGTHS IN DECIMALS OF MM.	NO. OF WAVES IN 1 MM. OF LENGTH.	NO. OF VIBRATIONS IN 1 SECOND. MILLIONS OF MILLIONS.	ABSOLUTE REFRACTION INDICES. μ.
A.......	0.00076339	1310	393.	1.60756
a........	0.00071895	1391	417.3	1.60926
B.......	0.00068748	1455	436.5	1.61079
C.......	0.00065655	1523	456.9	1.61252
D.......	0.00058956	1696	508.8	1.61728
E......	0.00052704	1897	569.1	1.62353
b........	0.00051860	1928	578.4	1.62495
F.......	0.00048640	2056	616.8	1.62917
G.......	0.00043113	2319	695.7	1.64006
H.......	0.00039714	2518	755.4	1.64969

390. ***Polarized Light.*** While we have assumed in Part I that the vibrations of the molecules of the medium which transmit the undulations of light are transversal, nothing in the subsequent discussion of light has so far indicated the importance of this assumption. And were it not for the phenomena cf polarization, we might with equal indifference have assumed these vibrations to be in any direction with respect to the luminous ray. Any oblique vibration, however, can be resolved into its components, one longitudinal and the other transversal, and the question of direction is necessarily limited to a choice between them, provided the properties of polarized light demand a decision in this respect. By direct observation we have established that the sound waves which reach the ear are due to the *longitudinal* vibrations of the air, whether the sounding body itself vibrates transversely or longitudinally. But the waves of light are so minute, that we can never hope to discern by direct means the infinitesimal motions of the molecular ether. Nevertheless the marked differences in the effects produced by the superposition of two beams of natural light and by two of polarized light, leave no alternative but the acceptance of the *principle of transversal vibrations* as a real cause of all optical phenomena.

391. The theoretical and practical study of polarized light are quite separate and distinct, and either may take precedence of the

other. But as there are certain advantages gained by clearly appreciating the practical differences that exist between natural and polarized light, we will first briefly refer to some of the more prominent characteristics of the latter.

Theoretically, light is said to be *polarized* when the molecular vibrations of the ray are rectilinear and perpendicular to a plane called the *plane of polarization*. The operation of polarization consists in bringing all the molecular vibrations into this position, or by stopping out those that do not have this direction. When this condition is exactly fulfilled, the light is said to be *wholly polarized* or *plane polarized*; when otherwise it is said to be more or less *partially polarized*, according to the degree of its modification. Practically we define polarized light to be that which possesses certain peculiarities, hereafter to be described, by which it may readily be distinguished from natural light by the application of certain tests. To the unassisted eye there is no distinction between natural and polarized light, and we now proceed to indicate in what manner these peculiarities can be manifested.

392. Any apparatus used to change natural into polarized light is called a *polarizer*, and that which is used to ascertain whether light be polarized or not is called an *analyzer*. These are essentially the same and are perfectly interchangeable, deriving their names only from the particular use to which they are put.

393. The usual methods by which common or natural light is polarized, are *double refraction*, *reflection*, and *refraction*.

Polarization by Double Refraction. 1°. *Iceland Spar.* The *usual form* of a natural crystal of Iceland spar is that of an oblique rhombohedron, whose adjacent faces are unequal. The oblique and acute angles of the faces are 101° 55′ and 78° 5′ respectively. The *regular form* of the crystal is obtained by cleavage, and it is in this form when all the edges are of equal length. The right line joining the two obtuse solid angles when the crystal is in the regular form is called the *optic axis* of the crystal. The plane normal to any face, containing the direction of the axis, is the *principal plane*. If a crystal of Iceland spar be placed in the track of a beam of solar light in any direction except that of the optic axis, a separation of the beam into two components of equal intensity will take place, and these two beams will emerge from the

opposite face parallel to each other and to their original direction. This property is called *double refraction.*

Let us suppose a crystal of the regular form to be so mounted that its principal plane, at first vertical, may pass by all azimuths around the direction of a ray. We will notice that a horizontal incident beam of solar light will always be separated into two beams, one of which always follows the ordinary laws of refraction, Art. 79, and which for this reason is called the *ordinary* beam, while the other is not always so controlled, and is therefore called the *extraordinary* beam; these will be represented hereafter by the letters O and E. On rotating the crystal we will find that in every position of the principal plane these two beams will have equal intensities, and both will always lie in the principal plane.

Now if a second crystal exactly like the first be similarly placed near it, and the beam E be stopped out by a diaphragm after it emerges from the first crystal, we will notice the following phenomena: Calling α the azimuth angle between the principal planes of the crystals, and supposing the second crystal only to rotate, when $\alpha = 0$ there will be but one beam passing the second crystal, whose displacement is double that due to either crystal; as α increases from 0 to 45°, two beams of unequal intensity will pass the second crystal, which we will designate by O_o and O_e. O_o, at first of greater intensity, will gradually diminish, while O_e will increase, both being equal when α is 45°; as α increases from 45° to 90°, O_o will decrease to extinction, and O_e will increase until it becomes as bright as O_o was formerly; as α increases from 90° to 180°, similar changes will occur, so that O_o will increase to a maximum, and O_e decrease to extinction; as α increases from 180° to 270°, the changes of the first quadrant will recur, and from 270° to 360° those of the second quadrant will be reproduced. If now O be stopped at emergence from the first crystal, and E be permitted to pass, and we represent its components by E_o and E_e, we will find, on rotating the second crystal, that E_o will be zero at 0° and 180°, and E_e a maximum; at 90° and 270°, E_o will be a maximum and E_e zero; at intermediate azimuths, E_o and E_e will differ in intensity, except at 45°, 135°, 225°, and 315°, where they will be equal.

394. In these results which are arranged below, I represents the intensity of the original beam, and α the angle included between the principal planes of the two crystals:

$$I = \begin{cases} O = \frac{1}{2}I = \begin{cases} O_o = \frac{1}{2}I\cos^2\alpha, \\ O_e = \frac{1}{2}I\sin^2\alpha, \end{cases} \\ E = \frac{1}{2}I = \begin{cases} E_o = \frac{1}{2}I\sin^2\alpha, \\ E_e = \frac{1}{2}I\cos^2\alpha. \end{cases} \end{cases} \tag{467}$$

Neither O nor E is similar to natural light, for when either is received on the second crystal, two emergent beams do not *always* result; with natural light, two beams are always produced. Neither are O and E similar to each other, for in certain positions of the second crystal, O produces only an ordinary beam, and E only an extraordinary beam; in certain other positions, O produces only an extraordinary beam, and E only an ordinary beam. They therefore have peculiar properties depending on the position of the crystal. But we see that O and E have properties of the same kind with reference to two planes passing through their direction, the first being the principal plane and the second at 90° from it. For these reasons the beams O and E, and their components O_o, O_e, E_o, E_e, are said to be *polarized*, the O's in the plane of principal section, and the E's in the plane at right angles to the principal section.

395. 2°. *By a Plate of Tourmalin.* All crystals which, like Iceland spar, have but one axis of symmetry, act in the same manner upon light. Tourmalin is such a crystal, and when cut into a thin plate *parallel* to the axis, can be used to polarize light. We should therefore have, 1°, an extraordinary beam E, equal to $\frac{1}{2}I$, but which by absorption is reduced to $\frac{1}{2}Iu^t$, polarized perpendicular to the principal section; and 2°, an ordinary beam O, polarized in the plane of principal section, and whose intensity should be $\frac{1}{2}Iu'^t$; u' is however so small, that for very little thicknesses the intensity of O that passes is practically negligible, and hence E emerges sensibly free from O. If two of these plates be superposed,

$$E = \frac{1}{2}Iu^{2t}\cos^2\alpha;$$

hence, when the axes are parallel,

$$E = \frac{1}{2}Iu^{2t},$$

and when perpendicular, E is zero.

396. 3°. *By a Nicol Prism.* If a crystal of Iceland spar, having equal width and thickness, and a length three times its

width, be cut along the plane containing the longer diagonal, and the surfaces be polished and reunited with a thin layer of Canada balsam, in their original position, the crystal is called a *Nicol prism*. The refractive index of the balsam, 1.549, lies between 1.654, that of O, and 1.483, that of E. Hence, an incident ray of natural light being separated into two rays O and E within the crystal, the former will meet the surface of the balsam at an angle of incidence such that it will be totally reflected to the side of the crystal, where it is absorbed, while the E ray will pass through unchanged. A layer of air serves the same purpose as the balsam, with the advantage of requiring a shorter length of crystal for the prism. The emergent ray E is polarized in the plane perpendicular to the principal section of the prism.

397. *By Reflection.* If a small beam of natural light be incident on a black glass mirror, the reflected light will be that due to reflection from the first surface. If the angle of incidence be about 54° 35′, and the reflected beam be examined *on all sides* by another mirror at the same angle of incidence, we will observe that when the planes of the two mirrors are parallel or perpendicular to each other, the intensity of the second reflected beam is a maximum; at positions 90° from these, the intensity is zero, and at intermediate positions the intensity is variable. The law which governs the variation in intensity is given by the equation

$$I' = I \cos^2 \alpha,$$

in which I' is the intensity of the second reflected beam at any azimuth, I the intensity of the same beam at the position corresponding to $\alpha = 0$, and α is the angle included between the planes of incidence of the two mirrors. The plane of first incidence is called the plane of polarization of the beam under examination.

398. *By Refraction.* If the beam of natural light be incident on a bundle of thin glass plates at the same angle, 54° 35′, and the transmitted beam be examined in the same way, it will be found to possess similar properties, except that the equation for intensity will be

$$I' = I \sin^2 \alpha,$$

and the plane of polarization will be perpendicular to the plane of first incidence.

By either of these tests, therefore, a beam of light may be examined, its properties in the above point of view ascertained, and it may then be classified either as natural light, or polarized light, either wholly or partially.

399. The differences between natural and wholly polarized light are thus contrasted:

	Natural Light.	Polarized Light.
Iceland spar.	Always two beams of *equal* intensity transmitted in *all* azimuths of the principal plane.	Always two beams of *varying* intensity transmitted, except in the azimuths 0°, 90°, 180°, and 270° of the principal plane, at which there will be but one beam. In 0° and 180°, if beam is *O*; and in 90° and 270°, if beam is *E*.
Tourmalin and Nicol Prism.	Always one beam of *uniform* intensity transmitted at all azimuths of the axis of the crystal.	Always one beam of *varying* intensity transmitted, except at azimuths 0° and 180°, if beam is *O*; and 90° and 270°, if beam is *E*.
Reflection by black glass at an incidence of 54° 35′.	Always a beam of *uniform* intensity reflected at all azimuths.	Always a beam of *varying* intensity reflected at all azimuths, except at 0° and 180°, if beam is *E*; and of 90° and 270°, if beam is *O*.
Refraction through a bundle of glass plates at an incidence of 54° 35′.	Always a beam of *uniform* intensity refracted through the bundle at all azimuths.	Always a beam of *varying* intensity refracted at all azimuths, except at 0° and 180° for *O*, and 90° and 270° for *E*.

400. Experiment shows that, when by any of these or other methods light polarized in one plane is produced from natural light, there is always produced at the same time light polarized in a perpendicular plane; and that light polarized in one plane cannot be made to give light polarized in a perpendicular plane. Experiment further shows that whenever two plane polarized beams are produced from the same source, and we derive from them by any means other plane polarized beams whose planes are coincident, interference takes place exactly as in the case of natural light; but

light polarized in one plane cannot be destroyed by light polarized in the perpendicular plane.

If light be due to longitudinal vibrations, no reason can be given why interference should not take place independently of the position of the planes of polarization of a ray; for the velocity of the molecule in the direction of the ray cannot be influenced in any way by any azimuthal change of the plane of polarization around the ray. If, however, we suppose the vibrations transversal, it is evident that any azimuthal change of one of the planes of polarization will give a component displacement in the direction of that due to the other, so that the former may be combined with the latter, and thus determine a resultant displacement of greater or less magnitude, depending on the difference of phase of the components.

401. ***Analytical Proof of the Transversality of the Molecular Vibrations.*** Let us consider a molecule of ether on the axis of x, along which a plane polarized ray is moving whose plane of polarization coincides with xy. Whatever be the rectilinear vibration of the molecule, the molecular displacements in the direction of the co-ordinate axes x, y, z, may be represented by

$$\left.\begin{aligned} u &= a \sin 2\pi \left(\frac{t}{\tau} - \frac{\phi}{\lambda}\right), \\ v &= b \sin 2\pi \left(\frac{t}{\tau} - \frac{\psi}{\lambda}\right), \\ w &= c \sin 2\pi \left(\frac{t}{\tau} - \frac{\chi}{\lambda}\right), \end{aligned}\right\} \qquad (468)$$

Suppose another plane polarized ray, having a different intensity and difference of phase δ at the time t, but the same plane of polarization also moving along the axis of x; the component displacements of the molecule will be

$$\left.\begin{aligned} u_{,} &= ma \sin 2\pi \left(\frac{t}{\tau} - \frac{\phi + \delta}{\lambda}\right), \\ v_{,} &= mb \sin 2\pi \left(\frac{t}{\tau} - \frac{\psi + \delta}{\lambda}\right), \\ w_{,} &= mc \sin 2\pi \left(\frac{t}{\tau} - \frac{\chi + \delta}{\lambda}\right). \end{aligned}\right\} \qquad (469)$$

Now let us suppose that the plane of polarization of the second ray turns about the axis of x until it is perpendicular to the plane of polarization of the first ray, and therefore coincides with the plane xz. After this revolution, the component displacement v, along y becomes the new component displacement w' along z, and the component displacement w, along z taken with a negative sign becomes the new component displacement v' along y; the component displacement u, along x remains unchanged in intensity and direction. The three components for the second plane polarized ray are then

$$\left.\begin{aligned} u' &= ma \sin 2\pi\left(\frac{t}{\tau} - \frac{\phi + \delta}{\lambda}\right), \\ v' &= -mc \sin 2\pi\left(\frac{t}{\tau} - \frac{\chi + \delta}{\lambda}\right), \\ w' &= mb \sin 2\pi\left(\frac{t}{\tau} - \frac{\psi + \delta}{\lambda}\right), \end{aligned}\right\} \tag{470}$$

Combining the displacements in the same directions by Eq. (61), then the sum of the resulting squares will be the measure of the resultant intensity due to the concurrent effect of the two plane polarized rays. By the application of the method of Art. 67, this resultant will be

$$\left.\begin{aligned} I &= a^2 + b^2 + c^2 + m^2a^2 + m^2b^2 + m^2c^2 + 2ma^2 \cos 2\pi\frac{\delta}{\lambda} \\ &\qquad - 2mbc \cos 2\pi\frac{\chi - \psi + \delta}{\lambda} + 2mbc \cos 2\pi\frac{\psi - \chi + \delta}{\lambda} \\ &= (a^2 + b^2 + c^2)(1 + m^2) + 2ma^2 \cos 2\pi\frac{\delta}{\lambda} \\ &\qquad + 4mbc \sin 2\pi\frac{\delta}{\lambda} \sin 2\pi\frac{\chi - \psi}{\lambda}. \end{aligned}\right\} \tag{471}$$

402. Since experiment shows that two perpendicular plane polarized waves can never interfere, this intensity is constant, whatever be the value of the difference of phase δ, which therefore exacts as simultaneous conditions that

$$a = 0 \quad \text{and} \quad bc \sin 2\pi\frac{\chi - \psi}{\lambda} = 0. \tag{472}$$

The first condition shows that the displacement in the direction of propagation is zero, and consequently that the *vibrations due to light are transversal.* The second condition can be satisfied in one of three ways, viz., either

$$b = 0, \quad \text{or} \quad c = 0, \quad \text{or} \quad \sin 2\pi \frac{\chi - \psi}{\lambda} = 0.$$

If we suppose either b or c is zero, the vibrations are rectilinear, and either parallel or perpendicular to the plane of polarization; both b and c cannot be supposed zero at the same time, without supposing both rays to be of zero intensity. The third supposition gives as a consequence,

$$\chi - \psi = n\frac{\lambda}{2},$$

n being any whole number, and therefore for the displacements in the direction of y and z for the first ray,

$$\left.\begin{aligned} v &= \quad b \sin 2\pi \left(\frac{t}{\tau} - \frac{\psi}{\lambda}\right), \\ w &= \pm c \sin 2\pi \left(\frac{t}{\tau} - \frac{\psi}{\lambda}\right), \end{aligned}\right\} \tag{473}$$

We have as a consequence of this supposition that the vibrations are rectilinear, and parallel to a right line in the plane yz, which is inclined to the axis of y by an angle whose tangent is $\pm \frac{c}{b}$. This conclusion is not admissible, because observation proves that there is a complete symmetry of all the phenomena of polarization with respect to the plane of polarization xy. This supposition must then be discarded, and either one of the other two adopted; whence we conclude that the molecular vibrations are always *transversal,* and in a plane polarized ray are either *parallel* or *perpendicular* to the plane of polarization. The best authority as well as the strongest experimental evidence are in favor of the latter conclusion, and therefore we will assume the direction of the molecular vibrations to be *perpendicular* to the plane of polarization. As the latter plane is readily determined, the particular direction of the molecular vibration follows as a consequence.

403. ***Distinction between Natural and Polarized Light.*** Let the direction of the plane polarized ray be along the axis of x, and the molecular vibration in the plane yz. Whatever be the direction and amount of the displacement, it may be resolved into its components along y and z, which will always bear a constant ratio to each other. Thus, if the displacement be

$$a \sin 2\pi \frac{t}{\tau}, \tag{474}$$

its components along y and z will be

$$\left.\begin{array}{l} a \cos \alpha \sin 2\pi \dfrac{t}{\tau}, \\[2ex] a \sin \alpha \sin 2\pi \dfrac{t}{\tau}, \end{array}\right\} \tag{475}$$

α being the angle which the displacement makes with y. The intensity of the ray will be a^2, and that of its components,

$$a^2 \cos^2 \alpha \quad \text{and} \quad a^2 \sin^2 \alpha.$$

Conversely, if we have two component displacements, having the same phase at the time t, along y and z,

$$\left.\begin{array}{l} b \sin 2\pi \dfrac{t}{\tau}, \\[2ex] c \sin 2\pi \dfrac{t}{\tau}, \end{array}\right\} \tag{476}$$

their resultant $d \sin 2\pi \dfrac{t}{\tau}$ will be definite, and its direction makes an angle whose tangent is $\dfrac{c}{b}$ with the axis of y.

404. Now since natural light is always decomposable, as shown by experiment, into two plane polarized beams of equal intensity, whose vibrations are in right lines perpendicular to each other, it would seem from the above that their recomposition should give a plane polarized beam whose molecular vibrations should be along a right line in the azimuth of 45°. Experiment shows that such is not the case, but that their recomposition gives natural light again. In a ray of natural light, the amplitudes and phases must of neces-

sity undergo abrupt changes after certain periods of regularity, which are insignificant when measured by sensation, but are great when compared with τ, and the vibrations may be sometimes along y, sometimes along z, and sometimes in intermediate directions. When along any right line, we may suppose the displacement replaced by its components, $a \cos \alpha$ and $a \sin \alpha$, which by recomposition would give the supposed displacement. But as the line along which this is made varies each instant, each pair of components would reproduce a displacement continually varying in azimuth; such are supposed to be the characteristics of natural light. But when we consider originally the vibration of the molecule in a polarized ray, along any right line, while its amplitude and phase may be variable, its direction of vibration must be constant, and any recomposition of its components will always give a constant azimuth angle, since the ratio of the components at any instant is the tangent of this angle. Therefore when we wish to exhibit the effects of interference between two polarized beams, they must be obtained from the decomposition of a single plane polarized beam, since all the perturbations that arise in these components coming from the common resultant, must themselves be common and concurrent in the components. The intensities and phases may be as variable as possible, but the ratios of these intensities and the difference of phase will be constant. The analogy in this respect is as complete as in the case of the interference of natural light, where it was shown that these phenomena are only exhibited by two beams which arise from a single origin, and are made to travel over routes which differ by an odd number of times $\frac{\lambda}{2}$ to the place of meeting.

405. ***Construction of the Refracted Ray.*** Resuming the discussion of plane waves of homogeneous light in different media, since the velocity of propagation varies directly as the square root of the elastic force developed by the displacement, Eq. (96), when the elastic forces are known we can readily determine the velocities in the three cases to be considered.

1°. *Isotropic Media.* In such media, $a = b = c$, and the wave surface becomes a sphere whose equation is

$$x^2 + y^2 + z^2 = a^2. \tag{477}$$

Let us suppose the incident medium in all cases to be air, and the velocity of light in it to be unity. To construct the refracted ray, let MM′ (Figure 94) be a plane deviating surface separating air from the isotropic medium considered; let SI be the incident ray, and with radii of a and *unity* describe circles IR and IA about I as a centre. Prolong SI to A, and from B, where the tangent at A intersects MM′, draw BR; IR will be the refracted ray and BR the refracted plane wave. For from the figure we have

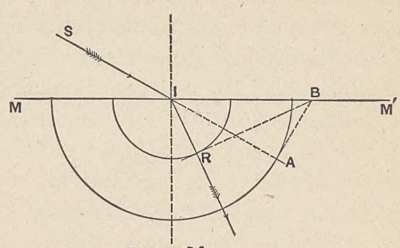

Figure 94.

$$\sin \mathrm{IBA} = \frac{1}{\mathrm{IB}} \quad \text{and} \quad \sin \mathrm{IBR} = \frac{a}{\mathrm{IB}}; \tag{478}$$

$$\therefore \quad \sin \mathrm{IBA} = \frac{1}{a} \sin \mathrm{IBR}, \quad \text{or} \quad \sin \phi = \mu \sin \phi'. \tag{479}$$

Hence, in all isotropic media it is readily seen that the refracted ray obeys both of the laws of Descartes.

406. 2°. *Uniaxal Crystals.* Let us suppose $b = c$; then, Art. 110, the wave surface becomes a surface of two nappes, one being a sphere with radius b, and the other an ellipsoid of revolution about the axis x, given by the equations,

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= b^2, \\ a^2x^2 + b^2(y^2 + z^2) &= a^2b^2. \end{aligned}\right\} \tag{480}$$

In every direction, except along the axis of x, there will be a separation of a single incident ray into two refracted rays, each moving with different velocities. The axis of x is called the *optic axis* of the crystal, and because of this one direction of single ray velocity, these crystals are called *uniaxal.* Whenever the plane of incidence is parallel or perpendicular to the principal section, the two refracted rays will be found in the plane of incidence, and the first law of Descartes will be obeyed by both rays; in every other plane of incidence, one of the refracted rays will lie outside of the

plane of incidence and obeys neither of these laws. In the first case, the construction of the refracted rays is a problem of Plane Geometry, and in the second of Solid Geometry. Both refracted rays are plane polarized, in planes at right angles to each other; that one which is polarized in the plane of principal section is the ordinary, and the other is the extraordinary ray.

407. To illustrate the construction in the simpler cases, let the *plane of incidence contain the optic axis* PP′, Fig. 95, and let SI be the incident ray. From I as a centre, with radius unity, construct as before the circle IA, and with radius b the ordinary wave IO. About PP′ as an axis, construct the section of the extraordinary wave IE; it will be an ellipse, whose semi-axes are b and a. Drawing tangents from B to O and E, we have IO and IE, the refracted ordinary and extraordinary rays respectively, and for the plane waves BO and BE respectively.

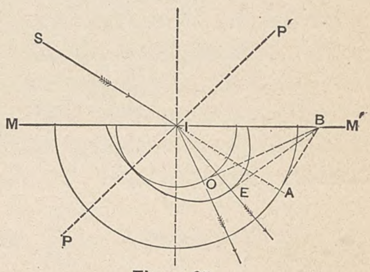

Figure 95.

408. Let the *plane of incidence be perpendicular to the optic axis*. Then, in Fig. 96, the sections of the wave surface are circles with radii b and a respectively, and both O and E obey both laws of Descartes; but the index of refraction of E is here $\frac{1}{a}$, which is less than $\frac{1}{b}$, or that of O. The index of refraction of E varies between its maximum $\frac{1}{b}$, and its minimum $\frac{1}{a}$, depending upon the position of the incident ray; but since when a is known its value for any particular incidence can be readily found, the extraordinary index is generally designated by $\frac{1}{a}$.

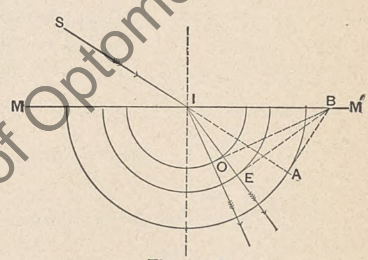

Figure 96.

409. Let the *optic axis lie in the plane of incidence and in the deviating surface;* then the construction is shown in Fig. 97. In this, as well as in the preceding cases, the ordinary ray is nearer the optic axis than the extraordinary ray. In all uniaxal crystals in which the spheroid is oblate, the ordinary ray will lie nearer the optic axis than the extraordinary ray, and they are called *negative* crystals; in all other uniaxal crystals, in which $b > a$, and hence the spheroid becomes prolate, the contrary will be the case, and such crystals are called *positive* crystals.

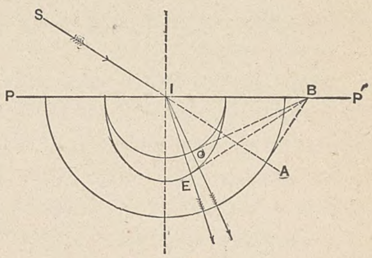

Figure 97.

410. 3°. *Biaxal Crystals.* These comprise all crystals in which the elasticities differ in three rectangular directions. The wave surface is that given by Eq. 146, whose more important properties have been deduced in Part I. When the plane of incidence is normal to one of the three axes of elasticity, one of the sections of the surface by this plane is a circle, and hence one of the rays follows the law of the sines; since three such planes may be passed, the three indices of refraction, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, so obtained are called the *principal* indices of the crystal. They can be experimentally determined by the method of minimum deviation, by means of three prisms cut parallel to the three axes of the crystal, and when once obtained suffice to define the optical qualities of the crystal.

411. ***Interior Conical Refraction.*** Possibly the most remarkable example of the complete accord of the predictions of theory with the results of experiment is the deduction of Sir William Hamilton, from a mathematical study of the properties of Fresnel's wave surface. It is seen (Fig. 18) that the tangent planes MN, etc., are each tangent to the wave surface along the circumference of a small circle, parallel to the circular sections of the ellipsoid, Eq. (180), and hence it follows that a single ray of natural light, incident externally in a direction such that after refraction it coincides with the optic axis OM, it should be separated into an

infinite number of rays within the crystal, taking the form of a hollow cone. Therefore, should the emergent face of the crystal be parallel to the incident face, this cone of rays should emerge as a hollow cylinder of rays, parallel to the original incident ray.

412. ***Exterior Conical Refraction.*** Again, when any two rays pass into a biaxal crystal in a single direction, their velocities are in general different, and are represented by the radii-vectores of the wave surface, while their directions on emergence are determined by the positions of the tangent planes to the two nappes of the wave surface at their points of intersection. But in the case of the ray OI, the two radii-vectores unite, and the rays have the same velocity. There are, however, an infinite number of tangent planes to the wave surface at the point I, and therefore a single ray of natural light, taking the direction IO within the crystal, should emerge into the air as a cone of rays, whose directions and planes of polarization depend on the position of the tangent planes.

Dr. Humphrey Lloyd completely established the truth of these predictions, by experiment upon the biaxal crystal arragonite, and thus furnished one of the most unexpected proofs of the general correctness of the basis upon which Fresnel had established the laws of double refraction. Fig. 98 illustrates the first case, or that of interior conical refraction, and Fig. 99 that of exterior conical refraction.

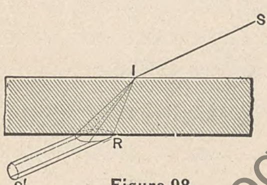

Figure 98.

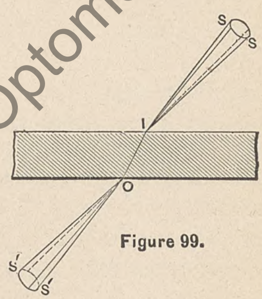

Figure 99.

413. ***Mechanical Theory of Reflection and Refraction.*** Fresnel, to whom we owe the solution of the problem of determining the *intensity* and *position* of the plane of polarization of reflected and refracted light, has based his analysis upon the following suppositions, viz.:

1°. That the motions of the molecular ether in and near the surface of separation of two media are the same in each, and therefore that the velocities parallel to the separating surface are equal.

2°. That the kinetic energy in the incident wave is equal to the sum of the kinetic energies of the reflected and refracted waves.

3°. That the elasticity of the ether in and near the surface of separation is the same for both media, and therefore that the densities of the ether molecules in the two media are directly as the squares of the refractive indices.

Although there is no sufficient reason *a priori* why these suppositions should be accepted unquestioned, the experimental verifications of the deductions which flow from them undoubtedly furnish ample grounds *a posteriori* of the correctness of the assumptions. Several of the most important laws, which had before been reached by experiment, are shown to be the direct consequences of the theory.

414. 1°. *Reflection of Light polarized in the Plane of Incidence.* The motion of the molecules are in this case normal to the plane of incidence. Let u and v represent the velocities of the molecules in the reflected and refracted rays respectively, and let the velocity of the molecules in the incident wave be unity. By the first principle we have

$$1 + u = v; \tag{481}$$

and by the second, $$m\,(1)^2 = mu^2 + m'v^2; \tag{482}$$

whence, $$u = \frac{m - m'}{m + m'}, \qquad v = \frac{2m}{m + m'}; \tag{483}$$

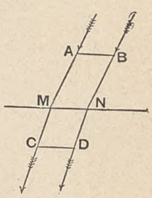

Figure 100.

which are identical with the expressions for the velocities of two perfectly elastic balls of mass m and m' after impact, unity being the velocity of m before impact. Let AM and MC (Fig. 100) represent the velocities and directions of the incident and refracted rays in the two media. The corresponding volumes of the ether put in motion by them in the two media are directly as the areas MB and MD; MB being that for the incident and also for the reflected wave, and MD that for the refracted wave. Or, if ϕ and ϕ'

represent the angles of incidence and refraction, these volumes are directly as

$$\sin\phi\cos\phi \quad \text{and} \quad \sin\phi'\cos\phi'.$$

By the third supposition, the densities of the ether of the two media are as

$$1 : \mu^2, \quad \text{or as} \quad \sin^2\phi' : \sin^2\phi.$$

Whence, for the ratio of the masses, we have

$$\sin\phi\cos\phi\sin^2\phi' : \sin\phi'\cos\phi'\sin^2\phi :: m : m';$$

or

$$\frac{m'}{m} = \frac{\tan\phi}{\tan\phi'}. \tag{484}$$

Substituting this value in Eq. (482), and eliminating v by Eq. (481), we have

$$\frac{1-u}{1+u} = \frac{\tan\phi}{\tan\phi'}; \tag{485}$$

whence,

$$\left.\begin{aligned} u &= -\frac{\sin(\phi-\phi')}{\sin(\phi+\phi')}, \\ v = 1 + u &= \frac{2\cos\phi\sin\phi'}{\sin(\phi+\phi')}. \end{aligned}\right\} \tag{486}$$

415. 2°. *Reflection of Light polarized Perpendicularly to the Plane of Incidence.* In this case, since the molecular vibrations are all in the plane of incidence, and are besides normal to the directions of the rays themselves, we have, representing by u' and v' the molecular velocities in the reflected and refracted waves,

$$\left.\begin{aligned} (1+u')\cos\phi &= v'\cos\phi', \\ m(1)^2 &= mu'^2 + m'v'^2; \end{aligned}\right\} \tag{487}$$

whence,

$$\frac{1-u'}{1+u'} = \frac{\sin\phi\cos\phi}{\sin\phi'\cos\phi'}; \tag{488}$$

and therefore,

$$\left.\begin{aligned} u' &= -\frac{\tan(\phi-\phi')}{\tan(\phi+\phi')}, \\ v' &= \frac{2\sin\phi'\cos\phi}{\sin(\phi+\phi')\cos(\phi-\phi')} \end{aligned}\right\} \tag{489}$$

416. Let the intensity of the incident light in the two cases be represented by *unity;* then the intensities of the reflected and refracted components will be u^2 and $1 - u^2$, and u'^2 and $1 - u'^2$, respectively. Developing the values of u and u', and replacing $\sin\phi'$ and $\cos\phi'$ by their values in terms of ϕ and μ, we readily obtain

$$u^2 = \frac{\sin^2(\phi - \phi')}{\sin^2(\phi + \phi')} = \left(\frac{\sqrt{\mu^2 - \sin^2\phi} - \cos\phi}{\sqrt{\mu^2 - \sin^2\phi} + \cos\phi}\right)^2, \tag{490}$$

$$u'^2 = \frac{\tan^2(\phi - \phi')}{\tan^2(\phi + \phi')} = \left(\frac{\sqrt{\mu^2 - \sin^2\phi} - \mu^2\cos\phi}{\sqrt{\mu^2 - \sin^2\phi} + \mu^2\cos\phi}\right)^2, \tag{491}$$

from which we can obtain the values of u and u' for all values of ϕ. For $\phi = 0$, we have

$$u^2 = \left(\frac{\mu - 1}{\mu + 1}\right)^2 \quad \text{and} \quad u'^2 = \left(\frac{\mu - 1}{\mu + 1}\right)^2;$$

therefore, for normal incidence, the quantity of reflected light is the same in the two cases. As ϕ increases, u^2 will continually increase, until when $\phi = 90°$,

$$u^2 = 1;$$

whence the reflected light is equal to the incident, when the incident beam is polarized in the plane of incidence. But u'^2 does not increase continuously for an increase of ϕ; it decreases from $\left(\frac{\mu - 1}{\mu + 1}\right)^2$ for $\phi = 0$ to zero for $\phi = \tan^{-1}\mu$, and then increases to unity as ϕ increases to 90° from this angle. There will then be no reflected light when the incident light is polarized perpendicularly to the plane of incidence and the angle of incidence is $\tan^{-1}\mu$. We also see, since for $\phi = 0$,

$$u^2 = u'^2 = \left(\frac{\mu - 1}{\mu + 1}\right)^2,$$

that the intensity of the reflected light is at this incidence independent of the plane of polarization of the incident light.

417. 3°. *Reflection of Natural Light.* Natural light is produced by the vibratory motions taking place successively in every direction on the surface of the wave. Each of these motions may

be resolved into two others polarized in planes at right angles to each other. However these components may vary at every instant, the rapidity of these variations is such that in any appreciable time the sums of the component intensities are exactly equal in the two planes, and therefore we may regard a ray of natural light as being equivalent to two rays of equal intensity polarized in planes at right angles to each other. If we consider the component rays to be polarized, one in the plane of incidence and the other in a perpendicular plane, we may apply the conclusions deduced above to these separately, combine the results, and then find the intensities of the reflected and refracted components of the natural ray. We will therefore have, for the intensity of the reflected components,

$$\left.\begin{aligned} \frac{1}{2}u^2 &= \frac{1}{2}\cdot\frac{\sin^2(\phi-\phi')}{\sin^2(\phi+\phi')}, \\ \text{and} \qquad \frac{1}{2}u'^2 &= \frac{1}{2}\cdot\frac{\tan^2(\phi-\phi')}{\tan^2(\phi+\phi')}; \end{aligned}\right\} \tag{492}$$

and for the intensity of the reflected component of the natural ray,

$$\frac{1}{2}(u^2 + u'^2);$$

and similarly, for the intensity of the refracted component,

$$\frac{1}{2}(v^2 + v'^2) = \frac{1}{2}(1 - u^2) + \frac{1}{2}(1 - u'^2).$$

Since $u > u'$, we see that the reflected ray is not of the same nature as natural light, and for the same reason neither is the refracted ray. In the former there will be an excess of light polarized in the plane of incidence, which is measured by the difference of the component intensities, $\frac{1}{2}(u^2 - u'^2)$, and in the latter there will be an excess of polarized light in the plane perpendicular to the plane of incidence, measured by the same difference. We therefore see that the reflected and refracted rays of an incident natural ray contain *equal* quantities of polarized light, but polarized in planes at right angles to each other. This conclusion, which has been experimentally demonstrated by Arago, is therefore a consequence of Fresnel's theory. The operation of reflection and refraction then results in *partial* polarization of natural light.

The reflected light will be completely polarized when one of the portions of which it consists disappears. But the first portion which is polarized in the plane of incidence cannot disappear, for we have shown that the value of its intensity constantly increases as ϕ increases, from $\left(\frac{\mu - 1}{\mu + 1}\right)^2$ to unity when $\phi = 90°$. But the second portion vanishes when $(\phi + \phi') = 90°$; this relation gives

$$\cos \phi = \sin \phi' = \frac{\sin \phi}{\mu}, \quad \text{or} \quad \tan \phi = \mu.$$

This law, *that the tangent of the angle of maximum polarization is equal to the index of refraction of the medium,* was experimentally discovered by Brewster, and is another consequence of Fresnel's theory. For glass this angle is about 54° 35′, and is the same that was referred to in Art. 397, in polarization by reflection.

When $\phi + \phi' > 90°$, the expression for u' changes its sign, which is equivalent to a change of phase of 180° as the angle of incidence passes the polarizing angle.

418. ***Change of Plane of Polarization.*** If α represent the angle made by the plane of polarization of a plane polarized ray with the plane of incidence, we may suppose the vibration ω resolved into two components, *one* in the plane of incidence $\omega \cos \alpha$, and the *other,* $\omega \sin \alpha$, perpendicular to it. After reflection, these become

$$\left.\begin{aligned} & -\omega \cos \alpha \frac{\sin (\phi - \phi')}{\sin (\phi + \phi')}, \\ \text{and} \qquad & -\omega \sin \alpha \frac{\tan (\phi - \phi')}{\tan (\phi + \phi')}, \end{aligned}\right\} \qquad (493)$$

respectively. Calling α' the angle of the resultant vibration, we have

$$\tan \alpha' = + \tan \alpha \frac{\cos (\phi + \phi')}{\cos (\phi - \phi')}, \qquad (494)$$

from which the value of α' can be determined. If $\alpha = 0$, then $\alpha' = 0$; and if $\alpha = 90°$, $\alpha' = 90°$; therefore, when the plane of polarization of the incident ray either coincides with or is perpendicular to the plane of incidence, the plane of polarization of the

reflected ray is unchanged. When $\phi + \phi' = 90°$, then $\alpha = 0$, and the plane of polarization of the reflected ray coincides with the plane of incidence, whatever be the azimuth of the incident ray. In all other cases, the plane of polarization of the reflected ray approaches the plane of incidence, since in these cases $\alpha' < \alpha$. In a similar manner, it may be shown that the plane of polarization of the refracted ray *recedes* from the plane of incidence, the law of this recession being expressed by the formula

$$\cot \alpha' = \cot \alpha \cos (\phi - \phi'). \tag{495}$$

When the refracted ray meets a second surface parallel to the former and emerges into the air, the azimuth of the plane of polarization of the emergent ray is given by

$$\cot \alpha'' = \cot \alpha' \cos (\phi' - \phi) = \cot \alpha \cos^2 (\phi - \phi'). \tag{496}$$

The methods of complete polarization by successive reflection and refraction are therefore simple consequences of Fresnel's theory.

419. ***Elliptic Polarization.***—Let us suppose the direction of two plane polarized rays to be along the axis of x and their planes of polarization to be at right angles to each other, and their phases when they reach any molecule to differ. Then

$$y = b \sin \left(2\pi \frac{t}{\tau} - \beta\right), \qquad z = c \sin \left(2\pi \frac{t}{\tau} - \gamma\right), \tag{497}$$

will represent their amplitudes at any time t. From these we have

$$\sin^{-1} \frac{y}{b} = 2\pi \frac{t}{\tau} - \beta, \quad \text{and} \quad \sin^{-1} \frac{z}{c} = 2\pi \frac{t}{\tau} - \gamma; \tag{498}$$

whence $$(\beta - \gamma) = \sin^{-1} \frac{z}{c} - \sin^{-1} \frac{y}{b}.$$

Taking the cosines of both members and reducing we have

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - 2 \frac{yz}{bc} \cos (\beta - \gamma) = \sin^2 (\beta - \gamma); \tag{499}$$

whence we see that the molecular orbit is an ellipse referred to its centre and a pair of rectangular co-ordinate axes. If the compo-

nent rays are in the same phase or in phases differing by 180°, we have

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} \pm 2\frac{zy}{bc} = 0; \tag{500}$$

whence

$$\frac{y}{b} = \pm \frac{z}{c}, \tag{501}$$

or the orbit is a right line passing through the place of rest of the molecule, and the resultant ray is plane polarized.

When the component rays differ in phase by 90° and the vibrations are of equal amplitude, the orbit becomes a circle given by

$$y^2 + z^2 = b^2. \tag{502}$$

The polarization in these cases is said to be *elliptical, plane,* and *circular* respectively. In elliptical polarization the direction of the axes of the ellipse and the ratio of their lengths can be determined by observation. Thus, when such a beam is analyzed by a double refracting crystal whose principal section coincides with one of the axes of the ellipse, the component plane polarized beams have their maximum and minimum intensity; and when the plane of principal section is inclined to either axis by an angle of 45°, the two components are of equal intensity.

When a plane polarized ray undergoes reflection, the reflected ray is generally elliptically polarized; for the incident plane polarized ray may be supposed resolved into two rays, polarized respectively in the plane of incidence and in the perpendicular plane; the effect of reflection changing in general the phase of the vibration of the components unequally, when these are united the resultant ray is in the condition of that discussed above; that is, their vibrations are rectangular and differ in phase on the molecule at the place of meeting.

420. ***Reflection of a Polarized Ray at a Surface separating a Denser from a Rarer Medium.*** Resuming the expressions for the reflected molecular velocities for rays polarized *in* and *perpendicular to* the plane of incidence,

$$u = -\frac{\sin(\phi - \phi')}{\sin(\phi + \phi')}, \text{ and } u' = -\frac{\tan(\phi - \phi')}{\tan(\phi + \phi')}; \tag{503}$$

and considering the incident ray in the denser medium, we see that when $\phi' = 90°$ both u and u' become equal to unity. In this case, since there is no refracted ray, the intensity of the reflected ray should be equal to that of the incident. But when ϕ has greater values than that corresponding to $\phi' = 90°$, or that of total reflection, these expressions become, by substituting $\mu \sin \phi$ for $\sin \phi'$ and $\sqrt{-1} \cdot \sqrt{(\mu^2 \sin^2 \phi - 1)}$ for $\cos \phi'$.

$$\left.\begin{aligned} u &= \frac{\mu \sin \phi \cos \phi - \sin \phi \sqrt{-1} \cdot \sqrt{(\mu^2 \sin^2 \phi - 1)}}{\mu \sin \phi \cos \phi + \sin \phi \sqrt{-1} \cdot \sqrt{(\mu^2 \sin^2 \phi - 1)}} \\ u' &= -\frac{\sin \phi \cos \phi + \mu \sin \phi \sqrt{-1} \cdot \sqrt{(\mu^2 \sin^2 \phi - 1)}}{\sin \phi \cos \phi + \mu \sin \phi \sqrt{-1} \cdot \sqrt{(\mu^2 \sin^2 \phi - 1)}} \end{aligned}\right\} \qquad (504)$$

Placing $\qquad \tan \theta = \dfrac{\sqrt{(\mu^2 \sin^2 \phi - 1)}}{\mu \cos \phi}$

and $\qquad \tan \theta' = \dfrac{\mu \sqrt{(\mu^2 \sin^2 \phi - 1)}}{\cos \phi}$, $\qquad (505)$

the above expressions may be written

$$\left.\begin{aligned} u &= \cos 2\theta - \sqrt{-1} \cdot \sin 2\theta \\ u' &= \cos 2\theta' - \sqrt{-1} \cdot \sin 2\theta' \end{aligned}\right\}. \qquad (506)$$

421. These imaginary expressions suggest of themselves no reasonable interpretation. Fresnel, however, from the analogy of certain geometric cases in which the multiplication of the length of a line by the imaginary expression $\sqrt{-1}$ indicates a change of direction of the line equal to 90°, assumes in these cases that the phase of the vibration into which $\sqrt{-1}$ enters as a factor has undergone an increase of 90°, and therefore the expression

$$(\cos 2\theta - \sqrt{-1} \sin 2\theta) \sin \frac{2\pi}{\lambda} (vt - x) \qquad (507)$$

is to be interpreted as

$$\cos 2\theta \sin \frac{2\pi}{\lambda} (vt - x) - \sin 2\theta \sin \left[\frac{2\pi}{\lambda} (vt - x) + 90°\right], \qquad (508)$$

which may be written

$$\sin\left[\frac{2\pi}{\lambda}(vt - x) - 2\theta\right]. \qquad (509)$$

Similarly the other expression may be written

$$\sin\left[\frac{2\pi}{\lambda}(vt - x) - 2\theta'\right]. \qquad (510)$$

Representing the difference of phase of the component rays by δ, we have

$$\tan \tfrac{1}{2}\delta = \tan(\theta' - \theta) = \frac{\cos\phi \sqrt{(\mu^2 \sin^2\phi - 1)}}{\mu \sin^2\phi}, \qquad (511)$$

$$\cos\delta = \frac{1 - \tan^2 \frac{1}{2}\delta}{1 + \tan^2 \frac{1}{2}\delta} = \frac{2\mu^2 \sin^4\phi - (1 + \mu^2)\sin^2\phi + 1}{(1 + \mu^2)\sin^2\phi - 1}; \qquad (512)$$

whence $\delta = 0$ when $\sin\phi = \frac{1}{\mu}$ or $= 1$.

If we assume $\delta = 45°$, we have

$$\frac{2\mu^2}{(1 + \mu^2)\operatorname{cosec}^2\phi - \operatorname{cosec}^4\phi} = 1 + \sqrt{\tfrac{1}{2}}; \qquad (513)$$

and taking $\mu = 1.51$, the refractive index for crown glass,

$$\phi = 54° \, 37' \, 20'', \quad \text{or} \quad 48° \, 37' \, 30''.$$

422. Hence, if light be incident internally on the surface of crown glass at either of these angles, the phase of the vibrations in the plane of incidence is accelerated more than that of the vibrations perpendicular to this plane by 45°. And if light be twice reflected in the same plane of incidence, the difference of phase will be 90°. Fresnel's rhomb, Fig. 101, is constructed of crown glass, having the angles A and C each 54° 37′. Then light incident perpendicular to the face AB will be totally reflected at E and F and emerge normal to CD, or parallel to its original direction. The two reflections at E and F will accelerate the phases of the vibration parallel to the plane of incidence 90° more than those perpendicular to the same plane.

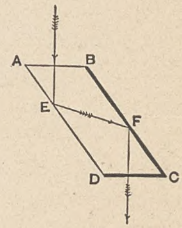

Figure 101.

423. Let α be the angle made by the plane of polarization of any incident plane polarized ray, with the plane of incidence in Fresnel's rhomb. Then the vibrations perpendicular to the plane of incidence are represented by

$$a \cos \alpha \sin \frac{2\pi t}{\tau}, \tag{514}$$

and those parallel to the plane of incidence, since they are accelerated by the angle δ, are

$$a \sin \alpha \sin \left(\frac{2\pi t}{\tau} + \delta\right). \tag{515}$$

Let y be the ordinate of the disturbed molecule in the plane of incidence, and z the ordinate in the perpendicular direction, both measured from the place of rest of the molecule. Then,

1°. If $\alpha = 45°$ and $\delta = 90°$, we have

$$y = a\sqrt{\tfrac{1}{2}} \cos \frac{2\pi t}{\tau}, \qquad z = a\sqrt{\tfrac{1}{2}} \sin \frac{2\pi t}{\tau}, \tag{516}$$

and

$$y^2 + z^2 = \frac{a^2}{2}; \tag{517}$$

hence the resulting ray is circularly polarized.

2°. If α have any value and $\delta = 90°$, we have

$$y = a \sin \alpha \cos \frac{2\pi t}{\tau}, \qquad z = a \cos \alpha \sin \frac{2\pi t}{\tau}, \tag{518}$$

and

$$\frac{y^2}{a^2 \sin^2 \alpha} + \frac{z^2}{a^2 \cos^2 \alpha} = 1; \tag{519}$$

hence the resulting ray is elliptically polarized, and the axes of the ellipse are $a \sin \alpha$ in the plane of incidence and $a \cos \alpha$ perpendicular to that plane.

3°. If α and δ have any values in general, we have

$$y = a \sin \alpha \left(\sin \frac{2\pi t}{\tau} \cos \delta + \cos \frac{2\pi t}{\tau} \sin \delta\right); \tag{520}$$

$$z = a \cos \alpha \sin \frac{2\pi t}{\tau}; \tag{521}$$

whence we have by combining and reducing,

$$(y - z \tan \alpha \cos \delta)^2 = a^2 \sin^2 \alpha \sin^2 \delta - z^2 \tan^2 \alpha \sin^2 \delta ; \quad (522)$$

which is the equation of an ellipse whose axes are inclined to the plane of reflection depending on α and δ.

4°. If $\alpha = 0$ or $\alpha = 90°$, the reflected ray retains its plane of polarization unchanged.

Therefore we see that by means of a Fresnel's rhomb, whenever the plane of polarization of the incident plane polarized ray makes angles with the plane of reflection of 0°, 90°, 180°, or 270°, the plane polarized light is not altered; when this angle is 45°, 135°, 225°, or 315°, the emergent light is circularly polarized, and for all other values of α the emergent light is elliptically polarized.

424. ***Circularly Polarized Light.*** If circularly polarized light be examined by an analyzer, it is evident that since it can be resolved into two equal vibrations parallel and perpendicular to any plane, it will present from this method of examination none of tne peculiarities of ordinary polarization; indeed it can not in this way be distinguished from natural light. Elliptically polarized light would, however, present different intensities on different sides, although in no position would the reflected ray reduce to zero intensity. If circularly polarized light be made to undergo two more total reflections in the same plane and at the same angle by transmitting it through a second rhomb placed parallel to the first, it will emerge plane polarized and its plane of polarization will be inclined 45° on the other side of the plane of incidence. Since the two additional reflections increase the difference of phase of the two portions from 90° to 180°, and the resultant of two equal vibrations at right angles to each other, whose phases differ by 180°, is a rectilinear vibration whose direction bisects the supplement of the angle formed by their directions.

If, in Fresnel's rhomb, $\alpha = -45°$, the transmitted light will also be circularly polarized; but in this latter case the motion of the molecule will be left-handed if that due to $\alpha = 45°$ was right-handed, and the reverse. Two such beams are said to be oppositely polarized, and the same distinction exists with respect to elliptically polarized light.

425. ***Interference of Polarized Light.*** If a plane polarized beam traverses a uniaxal crystal cut parallel to the optic axis of the crystal, it will in general be separated into two beams of unequal intensity. These components will have different velocities, and hence on emergence from the crystal will be in different phases. If the plate be thin and its incident and emergent faces parallel, the separation of the beams will be insensible. The two emergent beams being polarized in perpendicular planes can not interfere, but they may each be subdivided into two other plane polarized beams by means of an analyzer, having their planes of polarization coinciding with the principal section of the analyzer and at right angles to the principal section. One pair of these components that have the same plane of polarization will then conspire and the other pair will be in opposition; the particular color for which this interference takes place will vary according to the difference of route travelled in passing the crystalline plate, and therefore depends on the relation of its thickness to $\frac{\lambda}{2}$ for that color.

426. In Fig. 102, let PP be the trace of the plane of polarization of the incident beam on the face of the crystal, SS the trace of the principal section of the crystal, and AA the trace of the principal section of the analyzer on the same face; let α be the angle PCS, and β the angle PCA. Represent the amplitude of the vibration of the incident beam by unity; then the amplitudes of O and E will be represented on emergence from the crystal by cos α and sin α respectively. The components of O by the analyzer will be

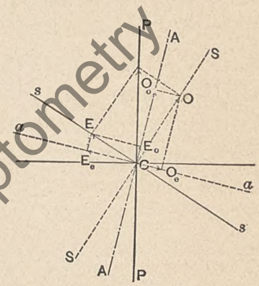

Figure 102

$$O_o = \cos \alpha \cos (\alpha - \beta),$$

and $$O_e = \cos \alpha \sin (\alpha - \beta),$$

and of E will be $$E_o = \sin \alpha \sin (\alpha - \beta),$$

and $$E_e = - \sin \alpha \cos (\alpha - \beta).$$

The diagram of the decomposition of the vibration in the figure will illustrate these values, if it be supposed turned in azimuth 90°

around C, since the vibrations are normal to the planes of polarization of the beams. The components O_o and E_o are polarized in the plane AA and form the ordinary emergent beam, while O_e and E_e are polarized in the plane *aa*, forming the emergent extraordinary beam. The beams O_o and E_o are not in the same phase, since the former has undergone ordinary refraction in the crystal and ordinary refraction in the analyzer, while the latter has undergone extraordinary refraction in the crystal and ordinary refraction in the analyzer; therefore, since the velocities of the ordinary and extraordinary ray are unequal, the two beams on leaving the analyzer will not in general be in accord. One will be retarded with respect to the other, and depending on the amount of this retardation, they may add or partially destroy each other's effects, or completely interfere. The whole of this difference is produced while the beams O and E are passing through the crystal, for in the analyzer both components considered, experience only ordinary refraction. Let d be the equivalent difference of route in air, due to the different velocities of O and E in passing through the crystal; then, from Eq. (76), we have for the intensity of the resultant of the pairs O_o and E_o, and O_e and E_e, the general expression

$$\left.\begin{aligned} \alpha^2 &= \alpha'^2 + \alpha''^2 + 2\alpha'\alpha'' \cos 2\pi \frac{d}{\lambda} \\ &= (\alpha' - \alpha'')^2 + 2\alpha'\alpha'' \left(1 + \cos 2\pi \frac{d}{\lambda}\right). \end{aligned}\right\} \qquad (523)$$

Replacing α^2, α'^2, α''^2, by I_o, O_o, E_o, and by I_e, O_e, E_e, we have, after reduction

$$I_o = \cos^2 \beta - \sin 2\alpha \sin 2(\alpha - \beta) \sin^2 \pi \frac{d}{\lambda}. \qquad (524)$$

$$I_e = \sin^2 \beta + \sin 2\alpha \sin 2(\alpha - \beta) \sin^2 \pi \frac{d}{\lambda}. \qquad (525)$$

427. These equations embody all the principles of interference of polarized light, and we readily deduce as consequences from them,

1°. That the *intensities* of the two beams are complementary, since $I_o + I_e$ is unity; and when the incident light is white, the *colors* of the two beams are complementary, since their superposition gives white light.

2°. That the two pencils are white, whatever be the retardation within the crystal, when

$$\sin 2\alpha \sin 2(\alpha - \beta) = 0;$$

therefore, when α is 0°, 90°, 180°, or 270°, or the principal section of the crystal is either parallel or perpendicular to the plane of polarization of the incident beam, and when $\alpha - \beta$ is 0°, 90°, 180°, or 270°, or the principal section of the analyzer is parallel or perpendicular to the principal section of the crystal.

3°. That the color is most intense when

$$\sin 2\alpha \sin 2(\alpha - \beta) = 1,$$

or when $\alpha = 45°$ and $\beta = 0°$ or $90°$; that is, when the principal section of the crystal is turned in azimuth 45° from the plane of primitive polarization, and the principal section of the analyzer is either parallel or perpendicular to the same plane.

4°. When α and β are unchanged, I_o and I_e vary only with $\frac{d}{\lambda}$; but d is proportional to the thickness of the crystal, its value being, for a given value of λ,

$$(\mu_o - \mu_e)\,t,$$

in which μ_o and μ_e are the principal indices of the crystal, and t the thickness. When t is increased so that d is increased by a whole number of times λ, the intensity resumes its original value. When d contains λ many times, or the crystal is thick, the colors become superposed on each other, and the resulting light is white, just as in the case of Newton's rings. The thicknesses between which these effects are displayed are in selenite, from $\frac{1}{2000}$ to $\frac{1}{50}$ of an inch, and similarly for other crystals.

428. ***Colored Rings in Uniaxal Crystals.*** Let ABDC, Fig. 103, be a uniaxal crystal cut *perpendicular* to the axis, and suppose a *converging cone* of polarized light be transmitted through it. We will also suppose that after it emerges from AB it is received by an analyzer, as in the previous case. Let PO be the direction of the axis. The ray PO will not be separated within the crystal, and emerges unchanged in polarization. If its plane of polarization is parallel to the principal section of the analyzer, it

will be transmitted, and if perpendicular it will not. All other rays will be modified, depending on the angle they make with the optic axis and the position of the principal section of the analyzer. Let the circle MN′M′ be the section of the emergent cone of rays, and MM′ and NN′ the traces of the plane of primitive polarization and the perpendicular plane respectively.

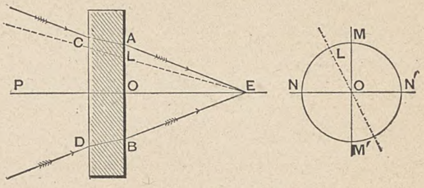

Figure 103.

All the rays which emerge on these lines will not be separated into two within the crystal, nor will their planes of polarization be altered. If the principal section of the analyzer coincides with either of these planes, the corresponding rays will be transmitted, and if perpendicular will not be transmitted. In the former case there will be a white cross, and in the latter a black cross. Rays which emerge at other points, such as L, will be separated into two within the crystal, whose planes of polarization will be parallel and perpendicular to OL, since such principal sections OL do not coincide with the primitive plane of polarization. The analyzer reduces each of these vibrations to the same plane, and interference will take place. The character of the interference will depend on the difference of phase, and hence on the interval of retardation.

429. The equation which expresses the law of the retardation can be obtained from Eq. (194), by making it applicable to uniaxal crystals. Thus, supposing $a > b$ and $b = c$,

$$u' = u'' = u;$$

and representing the velocity of the extraordinary ray by r, that of the ordinary being b, we have

$$\frac{1}{b^2} - \frac{1}{r^2} = \frac{a^2 - b^2}{a^2 b^2} \sin^2 u. \qquad (526)$$

In the case considered, the angle LEO is very nearly equal to u, and therefore all rays that meet the emergent face of the crystal, making $\sin u$ constant, will experience equal retardations. Taking all possible values of $\sin u$ as radii, we will have concentric rings of color, separated by dark rings surrounding O, and intercepted by

the bright or dark crosses referred to above, depending on the position of principal plane of the analyzer.

430. ***Biaxal Crystals.*** Let us suppose such a crystal cut perpendicularly to the *mean* line, that is, the line bisecting the angle made by the directions of equal wave velocity. The values of the retardations of ray velocities is given for these crystals by Eq. (194), or

$$\frac{1}{r'^2} - \frac{1}{r''^2} = \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin u' \sin u''. \qquad (527)$$

The curve corresponding to equal retardations will no longer be a circle, but the lemniscate, whose property is that the product of the radii-vectores drawn from two fixed poles to any point of the curve is a constant.

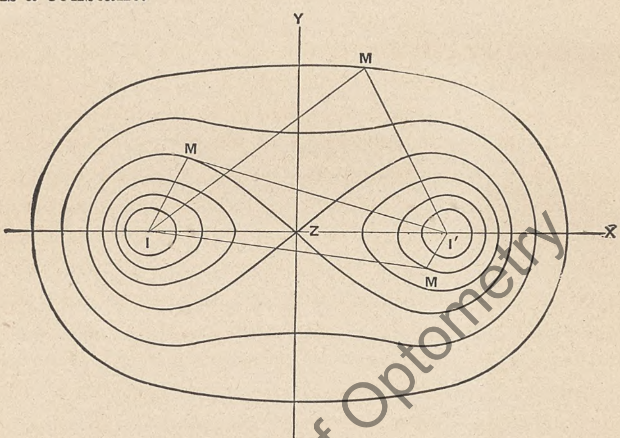

Figure 104

Figure 104 represents the system of curves of this character surrounding the poles I and I'. The directions of the vibrations of the two rays at any point M, M, etc., of either of these curves is readily seen, from Art. 130, to be in the bisectors of the angles formed by joining I and I' with the point M, and extending the lines beyond M. The form of the *dark brushes* which cross the entire system of rings is determined by the law which governs the planes of polarization of the emergent rays. In general, two such dark curves pass through each pole, which theory and experiment show to be hyperbolas having a common centre on the mean line.

431. Chromatic polarization furnishes valuable assistance to crystallography. It enables the observer to determine whether the crystal is uniaxal or biaxal, to find the positions and angles of inclination of its axes, and indeed almost to view the interior structure of the crystal itself. The properties of color by double refraction belong also to non-crystallized media when they have been subjected to changes by which their elasticities have been altered in certain directions. These changes may be brought about by mechanical pressure, or by sudden heating and cooling, and are exhibited in unannealed glass, or glass subjected to pressure or heat. The curves of color displayed when such substances are placed between the polarizer and analyzer are similar to those of crystals, and reveal the localities of equal wave retardation. Such phenomena belong to what is called *accidental double refraction.*

432. ***Rotatory Polarization.*** When a homogeneous ray of polarized light traverses a crystal of quartz in the direction of the axis, its plane of polarization is found to be turned about the ray from right to left, or the reverse, depending on the particular specimen employed. This phenomenon is called *rotatory polarization,* and the crystal is said to be right or left handed, according as the rotation is in the direction of or contrary to the motion of the hands of a watch as we look toward the crystal. It is due to the interference of the two opposite circularly polarized rays into which the plane polarized ray has been separated in its passage through the crystal. It is easily seen from what precedes that a plane polarized ray is equivalent to two circularly polarized rays of half the intensity, in which the vibrations are in opposite directions. Each is transmitted with a different velocity, and on emergence will unite into a plane polarized ray with its plane turned through an angle, due to the difference of phase of the components arising from the retardation which one of the components has experienced over the other. It is found that the rotation of the plane of polarization is different for different colors. Thus, for a plate of quartz 1 mm. in thickness, the angle of rotation of the extreme red was found to be $17\frac{1}{2}°$, and with an increasing value up to the violet, whose angle was $44°$.

433. When white light is used, each simple ray on emergence will have turned through an angle α, which is variable for each color; and if β be the angle which the principal section of the

analyzer makes with the plane of primitive polarization, the intensities of the O and E components of each simple ray will be proportional to $\cos^2(\alpha - \beta)$ and $\sin^2(\alpha - \beta)$, respectively. The tint formed by the mixture of these colors will be determined by these intensities, and hence, if a double refracting analyzer be used, the O and E components will be differently colored, but complementary. With a Nicol as an analyzer, among the tints which succeed each other as it is turned, there is one called the *tint of passage* or *transition* tint; it is a pale violet blue, and can always be easily recognized, for upon turning the Nicol the slightest degree, a clear blue is displayed in one direction, while in the contrary direction a vivid red is shown. By means of this tint the deviation of the plane of polarization can be found with the greatest precision.

434. While it seems natural to admit that the cause of rotatory polarization in quartz and other crystals in which this property exists is due to the regular arrangement of their molecules, this supposition fails to account for its existence in certain vapors and liquids, and therefore it is supposed to be due in the latter to the individual action exerted by the molecules themselves.

435. The application of rotatory molecular polarization has an important field in determining the state in which the molecules of an active substance is found, when dissolved in an inactive liquid; a state in which chemical analysis is powerless to discern. One of the most important practically, because of its commercial value, is its application to saccharimetry. The rotatory power of saccharine solutions furnishes a rapid and sure method of detecting the proportion of cane sugar which they contain, even when this sugar is mixed with other kinds whose rotatory power is different.

436. Our limits prevent a further allusion to these interesting subjects. Those which are of necessity omitted will be referred to in the accompanying lectures as far as time will permit. It is hoped that the slight insight here given will at least excite interest, and encourage subsequent study in those who may hereafter have the proper opportunities offered them. We have seen how varied and intricate are the phenomena, whose explanation flows from the simple fundamental principles of Mechanics, and can recognize the essential importance of gaining a more intimate knowledge of the mathematical processes by which these principles are able to expound the wonderful phenomena of natural science.

Date Due